Mathematical and Quantitative Methods

The Prediction of Exchange Rates with the Use of Auto-Regressive Integrated Moving-Average Models

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Abstract: Currency market is recently the largest world market during the existence of which there have been many theories regarding the prediction of the development of exchange rates based on macroeconomic, microeconomic, statistic and other models. The aim of this paper is to identify the adequate model for the prediction of non-stationary time series of exchange rates and then use this model to predict the trend of the development of European currencies against Euro. The uniqueness of this paper is in the fact that there are many expert studies dealing with the prediction of the currency pairs rates of the American dollar with other currency but there is only a limited number of scientific studies concerned with the long-term prediction of European currencies with the help of the integrated ARMA models even though the development of exchange rates has a crucial impact on all levels of economy and its prediction is an important indicator for individual countries, banks, companies and businessmen as well as for investors. The results of this study confirm that to predict the conditional variance and then to estimate the future values of exchange rates, it is adequate to use the ARIMA (1,1,1) model without constant, or ARIMA [(1,7),1,(1,7)] model, where in the long-term, the square root of the conditional variance inclines towards stable value.

Keywords: ADF; stationarity; ARIMA; EUR; prediction

JEL Classification: C32; C53; F31

1 Introduction: Literature Review

In today's global economy, the crucial importance for any future investments is the accuracy in predicting the foreign exchange rates or at least the correct prediction of the trend. There already are a great number of methods for predicting the exchange rates. It was shown by Robert Meese (MEESE R., 1983) that models based on the random walk hypothesis in predicting exchange rates are better than those based on macroeconomic indicators. However, this does not apply for the

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long-term prediction, which was proved by examining the prediction of USD exchange rate against four other currencies during seventeen years (MARK N., 1995).

Predictions of exchange rates made with the use of ARIMA models were started in nineties by economists Bellgard and Goldschmidt (BELLGARD C., 1999). However, they concluded that these models are not very suitable for predicting the exchange rates. Dunis and Huang (DUNIS C., 2002) who were using ARMA (4,4) were of the opposite opinion; their results were, however, insignificant.

Another example of a study using Box Jenkins models is for instance the paper "Exchange-rates forecasting: exponential smoothing techniques and ARIMA models", in which the authors investigated the behavior of daily exchange rates of the Romanian Leu against Euro, United States Dollar, British Pound, Japanese Yen, Chinese Renminbi and the Russian Ruble (FĂT M., 2011).

Weisang and Awazu (WEISANG G., 2008) presented three ARIMA models which used macroeconomic indicators to model the USD/EUR exchange rate. They discovered that over the time period from January 1994 to October 2007, the monthly USD/EUR exchange rate was best modeled by a linear relationship between its preceding three values and the current value. These authors also concluded that ARIMA (1,1,1) is the most suitable model for the prediction of the time series of USD/EUR exchange rate.

Another often used method for predicting the trend of exchange rates is the ANN model (Artificial Neural Network). Kamruzzaman J. a Ruhul A. Sarker (KAMRUZZAMAN J., 2003) developed and investigated three ANN based forecasting models using Standard Backpropagation (SBP), Scaled Conjugate Gradient (SCG) and Backpropagation with Baysian Regularization (BPR) for Australian Foreign Exchange to predict six different currencies against Australian dollar.

One of the recent studies (ROUT M., 2013) uses the hybrid model combining an adaptive autoregressive moving average (ARMA) architecture and differential evolution (DE) based on training of its feed-forward and feed-back parameters. The results of the developed model are compared with other four competitive methods such as ARMA-particle swarm optimization (PSO), ARMA-ca t swarm optimization (CSO), ARMA-bacterial foraging optimization (BFO) and ARMA-forward backward least mean square (FBLMS). The derivative based ARMA-FBLMS forecasting model exhibits the least suitable prediction performance of the exchange rates. Compared to that, ARMA-DE exchange rate prediction model possesses superior short and long range prediction potentiality compared to others. Many studies are dealing with the prediction of USD/EUR, USD/YEN or

USD/RON. The originality of this paper lays in the prediction of EUR against other European currencies for the long-term time horizon (2014-2020).

2. Methodology

In this paper, there are models of time series of monthly exchange rates of the national currencies not including the common European currency, for the time period of 12/1998 to 12/2013. These currencies are of the Czech republic (CZK), Poland (PLN), Great Britain (GBP), Romania (RON), Sweden (SEK) and Hungary (HUF). The data were obtained from the ECB database and they contain values of the selling price of each currency, specifically the average value for each month at the foreign exchange market FOREX. In total, there are 180 observations in the time series. Countries that do not use Euro but have fixed exchange rate were not included in this prediction. This applies for Bulgaria (1EUR=1,95583BGN), Denmark (1EUR=7,46038DKK) and Lithuania (1EUR=3,4528LTL).

To obtain the adequate ARIMA (p, d, q) model, the series stationarity was tested by applying the ADF-Augmented Dickey-Fuller (DICKEY&FULLER, 1979) and PP-Phillips-Perron unit root tests (PHILLIPS P., 1988). ADF was performed for the scenario with a constant, without a constant and with a trend. The most suitable appears to be the model with a constant, the results of which are shown below in the table no. 1 for different currencies. The results of these tests regarding non-stationarity of the indices are the same, namely the series EUR/RON, EUR/SEK, EUR/GBP, EUR/HUF are non-stationary (the null hypothesis of the unit root existence cannot be rejected, i.e. it is not a stationary time series).

Table 1. Augmented Dickey-Fuller (ADF) test with a constant for the currencies CZK/SEK/GBP/PLN/HUF/RON

Results/currency	CZK	SEK	GBP	PLN	HUF	RON
Estimated value γ	-0,001	-0,03	-0,01	-0,06	-0,01	0,003
Test statistics: t	-0,36	-1,81	-0,75	-2,82	-0,22	0,96
Asymptotic p-value	0,56	0,38	0,39	0,06	0,61	0,91

Source: author (SW Gretl)

If we do not reject the null hypothesis and the given series is non-stationary, it is necessary to proceed to its transformation, as the Box Jenkins (AR, MA, ARMA or ARIMA) models are based on the time series stationarity, in the form of

$$Y_n = a_1Y_{n-1} + a_2Y_{n-2} + \ldots + a_p\, Y_{n-p} - b_{1n-1} - b_{2n-2} - \ldots b_{qn-q} +_n \quad (1)$$
$$(1-a_1L-a_2L^2 - \ldots a_pL^p)Y_n = (1-b_1L- \ldots - b_qL^q)_n \quad (2)$$
$$\phi(L)Y_n = \theta(L)\varepsilon_n \quad (3)$$

Where p is the order of the autoregressive part, while q is the order of the moving average part, and ε_n represents the white noise.

Validation of ARMA (p,q) models is based on minimizing the AIC (Akaik's information criterion) and BIC (Schwarz's information criterion) criteria, as well as on the verification of the correlation of the error terms of the model and finally on measuring the divergence from the normality of these values. If it is needed for the time series to have one differential operation to achieve stationarity, it is a I(1) series. Time series is I(n) in case it is to be differentiated for *n* times to achieve stationarity. Therefore, ARIMA (p,d,q) models are used for the non-stationary time series, specifically the autoregressive integrated average models, where d is the order of differentiation for the series to become stationary. Therefore the ARIMA (p,d,q) model may be rewritten as follows:

$$\phi(L)\,(1- L)^d\, Y_n = \theta(L)\varepsilon_n \quad (4)$$

where L is the lag operator and the order of differentiation is equal to:

$$\Delta^d Y_n = (1- L)^d\, Y_n \quad (5)$$

The identification of modeling the conditional mean value is based on the analysis of estimated autocorrelation and partial autocorrelation function (ACF, PACF). These estimations may be strongly inter-correlated, it is therefore recommended not to insist on unambiguous determination of the model order, but to try more models. We must not forget to carry out the verification, which is based on retrospective review of the assumptions imposed on the random errors. Given that financial data are very often characterized by high volatility, it is necessary to test the model for ARCH effect, i.e. presence of conditional heteroscedasticity. Regarding heteroscedasticity it is therefore a situation where the condition of finite and constant variance of random components is violated. The following model illustrates the conditional heteroscedasticity:

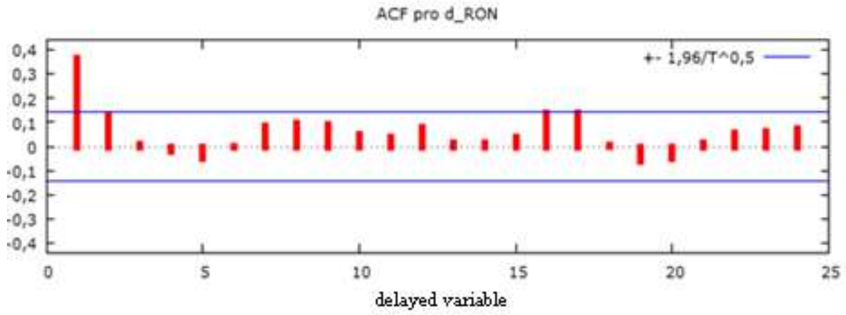
$$(lnX_t - lnX_{t-1})^2 = \alpha + \rho\,(lnX_{t-1} - lnX_{t-2})^2 + ut \quad (6)$$

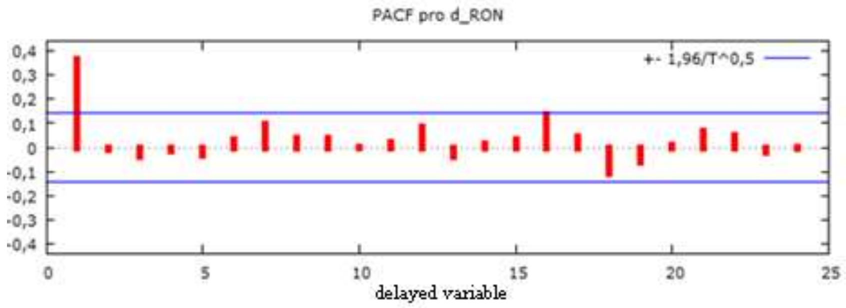
where Xt, Xt- represent values in the time series when time t is changed by one unit. The parameter α is calculated with the use of OLS and u_t is a random component. If the parameter ρ (regressive parameter) is equal to zero, we cannot talk about heteroscedasticity.

3. Results

Based on the priori information about the behavior of the exchange rates, it may be concluded that the specification of the ARIMA (1,1,1) type is an adequate choice. To verify this estimation, we generated the correlograms ACF and PACF which for most of the analyzed currencies confirm the legitimacy of the identification of the data generating process with the use of ARIMA (1,1,1). The exception is Swedish crown and Hungarian forint. Based on comparing the information criteria (AIC, BIC),ARIMA (1,1,1) model without constant was identified for the Swedish currency (SEK) and ARIMA [(1,7),1,(1,7)] model for the Hungarian currency.

Note. The model with a constant was also developed but compared to the significance of the p-value and by comparing the information criteria, it seems optimal to exclude the constant. For illustration, two ACF and PACF correlograms for the first difference for the Romanian Leu and Hungarian forint are shown below.





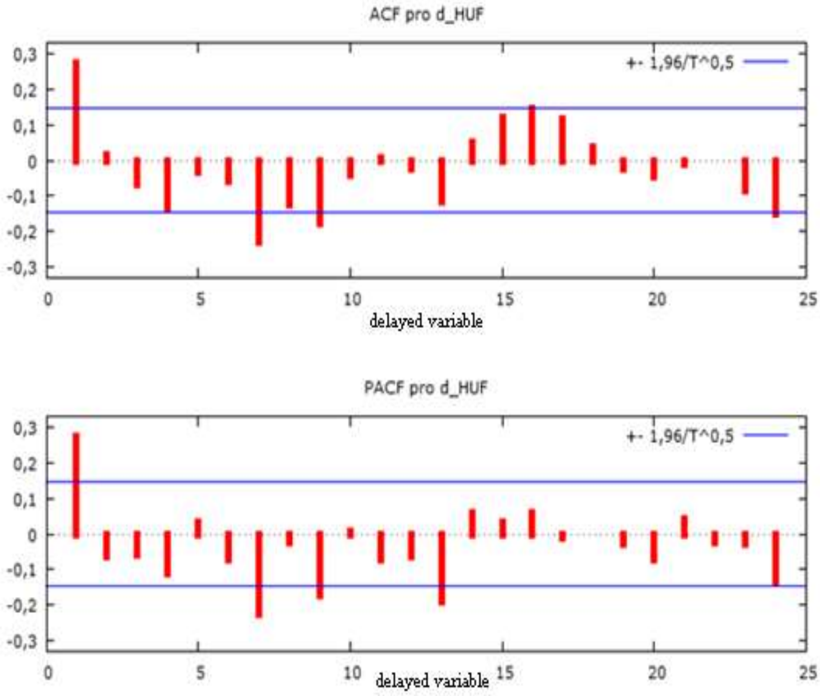


Figure 1. ACF and PACF correlograms for the first difference (RON, HUF)
Source: Author (SW Gretl)

CZK	coefficient	direct. error	z	p-value
phi_1	0,283049	0,0734204	3,855	0,0001***
theta_1	0,850331	0,0606456	14,02	1,15E-44***
SEK	coefficient	direct. error	z	p-value
phi_1	-0,527872	0,101123	-5,22	1,80e-07 ***
theta_1	0,850349	0,0606514	14,02	1,15e-44 ***
GBP	coefficient	direct. error	z	p-value
phi_1	0,186398	0,0736146	2,532	0,0113***
theta_1	-1	0,0233473	-42,83	0**
PLN	coefficient	direct. error	z	p-value
phi_1	0,391535	0,0691388	5,663	0,0000000149***
theta_1	-1	0,0148063	-67,54	0***
HUF	coefficient	direct. error	z	p-value
phi_1	0,342549	0,0850299	4,029	0,0000561***
phi_7	-0,21342	0,0702948	-3,036	0,0024***
theta_1	-1,09445	0,0527216	-20,76	1,02E-95***

theta_7	0,101755	0,0563949	1,804	0,0712*
RON	coefficient	direct. error	z	p-value
phi_1	0,346849	0,0752846	4,607	0,00000408***
theta_1	-0,972696	0,0248644	-39,12	0***

Figure 2. The estimation of the ARIMA for exchange rates with the use of 180 observations for the time period of 1.1999 – 31.12.1999 – 31.12.2013

Source: own calculations

Note. const = constant generated by SW Gretl, phi_1 = regressive coef. of AR processes at the 1. delay, theta_1: regressive coef. of MA processes at the 1. delay, z = test statistics.

From the table, we can conclude that parameters of AR member as well as of MA member are statistically significant at least on the 5 % level for all examined currencies. Then we tested the model for autocorrelation (H0: There is no autocorrelation in the model, H1: There is autocorrelation in the model). The result is the rejection of H1 in favor of null hypothesis, i.e. that there is no autocorrelation in the model, thus the chosen ARIMA (1,1,1) specification is adequate (eventually for HUF ARIMA [(1,7),1,(1,7)] model).

This is followed by testing the stationarity in the data generating process and finding, whether the model is invertible. This verification is based on discovering the absolute values of AR and MA roots.

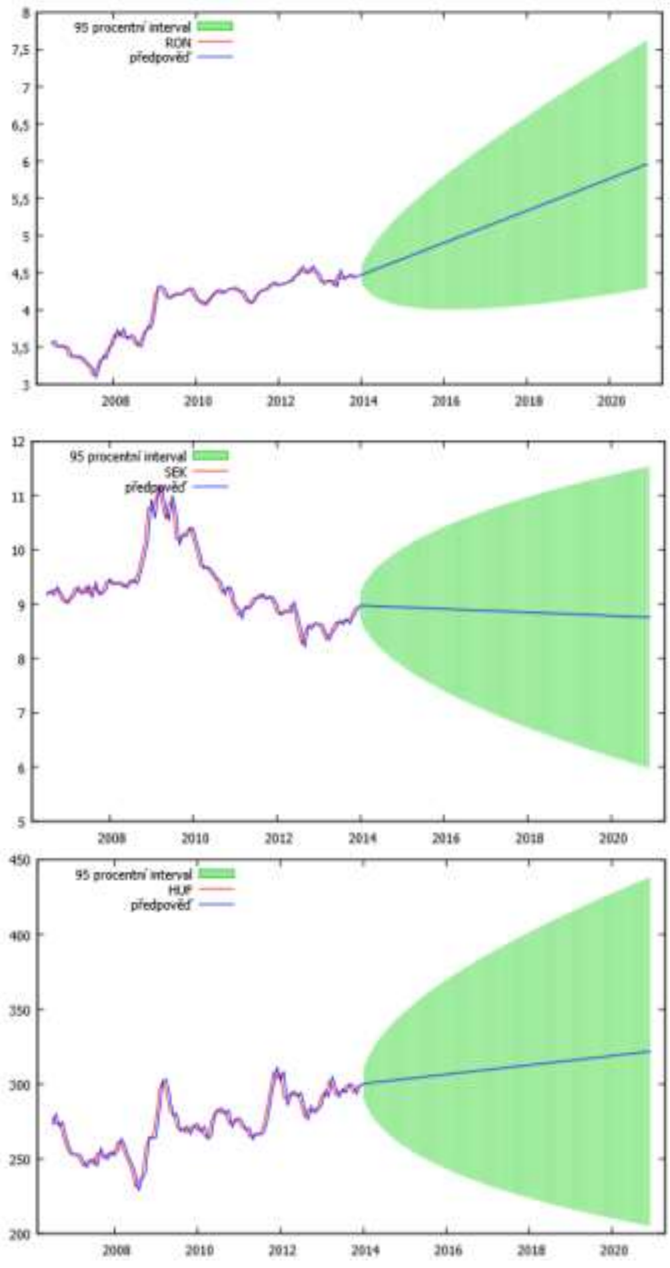
	CZK	SEK	GBP	PLN	HUF	RON
AR: Root 1 – abs. value	3,533	-1,8949	5,3649	2,5541	1,2963	2,8831
MA: Root 1 – absolute value	1,023	-1,176	1,001	1,005	1,0219	1,0281

Figure 3. Outputs of AR and MA roots

Source: author

Note. Hungary: absolute value of other six roots (in AR and MA) was higher than one.

The absolute value of all roots is higher than one, i.e. the model is stationary and invertible. Based on these results, we developed the prediction of exchange rates up to 2020, which is shown on the following pictures.


95 procentní interval
RON
předpověď
8
7,5
7
6,5
6
5,5
5
4,5
4
3,5
3
2008
2010
2012
2014
2016
2018
2020
95 procentní interval
SEK
předpověď
12
11
10
9
8
7
6
5
2008
2010
2012
2014
2016
2018
2020
95 procentní interval
HUF
předpověď
450
400
350
300
250
200
2008
2010
2012
2014
2016
2018
2020

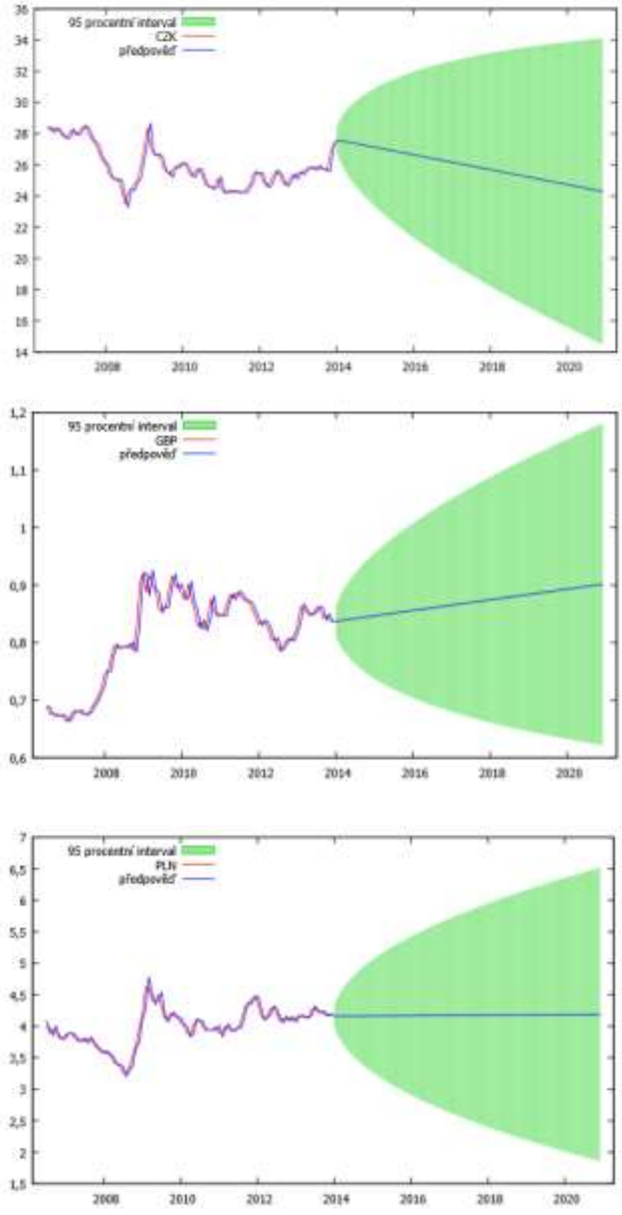


Figure 2. Prediction of exchange rates with the use of ARIMA model (RON, SEK, HUF, CZK, GBP, PLN)

Source: Author (SW Gretl)

Note: blue line = prediction, green line = 95% interval

The prediction shows decreasing trend of the Czech crown. Throughout the years, it should come to the evaluation of CZK against Euro but only slowly as according to the prediction the average exchange rate should be 24,52CZK/EUR in 2020. There is a slight decline also for the Swedish crown. For other currencies, the trend is rising or almost constant (Polish currency). From 2014 to 2020, Romania should register a rapid development of its currency. This rapid depreciation of the exchange rate (from 4,57RON/EUR in 2014 to 5,87RON/EUR in 2020) could cause pressure to increase the export by which the balance of payment deficit could be partly improved as Romania is currently overloaded with import.

Results of these models were subsequently verified with the use of the select autocorrelation function of standardized residues which verified their non-correlation. The final part of the verification was to test the normality of standardized residues and then, with the use of ARCH test, we determined, whether these residues have constant variance, i.e. whether they are conditionally homoscedastic. We tested the null hypothesis, i.e. that there is no ARCH effect present in the residues. If p-value is higher that the importance level of 0,05, we accept this hypothesis.

Currency	p-value	Result of the testing (presence of ARCH effect)
CZK	0,0348396	Residues are conditionally heteroscedastic
SEK	1,32E-05	Residues are conditionally heteroscedastic
GBP	0,00197651	Residues are conditionally heteroscedastic
PLN	0,11284	Residues are conditionally homoscedastic
HUF	0,098173	Residues are conditionally homoscedastic
RON	0,312715	Residues are conditionally homoscedastic

Figure 4. Detection of ARCH effect

Source: author

From the above shown table follows that heteroscedasticity was found at three currencies (i.e. presence of ARCH effect). Because the variances of the random components are not equal, the OLS method has not the optimal properties in this form of generalized linear regressive model, specifically it does not provide substantial estimations, however, these estimations are still impartial and may be used for further research. Polish, Hungarian and Romanian currency shows constant variance of the residues (homoscedasticity).

4. Conclusion

In this study, exchange rates of the six European currencies were predicted with the use of ARIMA (1,1,1) or with the use of ARIMA [(1,7),1,(1,7)]. The results are different for each selected currency – according to the prediction there will be appreciation as well as depreciation of the currency against Euro. It is however necessary to consider the limitations of using the ARIMA models which presented certain problems in estimating and validating the model and which are more effective in rendering the medium-term value (for several months). This long-term prediction should primarily show the future trend of the development of currencies exchange rates and at the same time identify the optimal model of the Box Jenkins models for predicting the European exchange rates. Both of these conditions were fulfilled.

5. References

Bellgard, C. & Goldschmidt, P. (1999). Forecasting across frequencies: Linearity and non-linearity. University of Western Australia Research Paper. *Proceedings of the International Conference on Advanced Technology*. Australia.

Dickey, D. A. & Fuller, W. (1979). Distribution of estimators for autoregressive time series with a unit root. *J. Am. Stat. Assoc*. 74, pp. 427–431.

Dunis, C. & Huang, X. (2002). *Forecasting and trading currency volatility: An application of recurrent neural regression and model combination*. England, Liverpool.

Făt, C.M. & Dezci, E. (2011). *Exchange-rates forecasting: exponential smoothing techniques and Arima models.* [online] Faculty of Economics and Business Administration, Department of Finance, “Babes-Bolyai” University, Cluj-Napoca, Romania, Retrieved from: http://steconomiceuoradea.ro/anale/volume/2011/n1/046.pdf.

Kamruzzaman, J. & Sarker, R. (2003). *Forecasting of Currency Exchange Rates using ANN: A Case Study*, [online] Gippsland School of Computing & IT School of Computer Science, Australia.

Mark, N. (1995). *Exchange rates and fundamentals: evidence on long-horizon predictability. American Economic Review,* pp. 201-218.

Meese, R. & Rogoff, K. (1983). *The out-of-sample failure of empirical exchange rates: sampling error or misspecification?* in FRENKEL, J. *Exchange Rates and International Macroeconomics,* University of Chicago Press, pp. 67-105.

Phillips, P. C. B. & Perron, P. (1988). *Testing for a unit root in time series regression*. Biometrika *75*, pp. 335–346.

Rout, M.; Majhi, B.; Majhi, R. & Ganapati, P. (2014). Forecasting of currency exchange rates using an adaptive ARMA model with differential evolution based training. [online] *Journal of King Saud University – Computer and Information Sciences* (2014) 26, pp. 7–18, Retrieved from http://ac.els-cdn.com/S1319157813000037/1-s2.0-S1319157813000037-main.pdf?_tid=5433f8e6-40a4-11e4-a1d8-00000aab0f6b&acdnat=1411203706_8fa301651e70ee218dbdca66a392f6d3.

Weisang, G. &Yukika, A. (2008). *Vagaries of the Euro: an Introduction to ARIMA Modeling*, [online] Bentley College, USA, Retrieved from http://www.bentley.edu/centers/sites/www.bentley.edu.centers/files/csbigs/weisang.pdf.

Forecast Intervals for Inflation Rate and Unemployment Rate in Romania

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Abstract: The main objective of this research is to construct forecast intervals for inflation and unemployment rate in Romania. Two types of techniques were employed: bootstrap technique (t-percentile method) and historical error technique (root mean square error method- RMSE). The forecast intervals based on point forecasts of National Bank of Romania (NBR) include more actual values of quarterly inflation rate during Q1:2011-Q4:2013. The proposed prediction intervals for quarterly inflation and unemployment rate contain the registered values. Considering as constant the error from previous year, we will build forecast intervals for annual inflation and unemployment rate based predictions provided by two anonymous experts on the horizon 2004-2015.All the forecast intervals for inflation rate based on first expert expectations included the actual values during 2004-2013.

Keywords: forecast intervals; point prediction; inflation rate; unemployment rate

JEL Classification: C51; C53

1 Introduction

The point forecasts did not provide any information regarding the degree of accuracy. On the other hand, the forecast intervals allow the evaluation of future uncertainty and the comparison between the forecasting methods, indicating the strategies to be applied for desired results.

The main aim of this paper is to construct forecast intervals for inflation and unemployment rate in Romania. Excepting some forecast intervals proposed by (Bratu, 2012, p. 146), in Romania prediction intervals for macroeconomic variables were not proposed. The most frequently used method for constructing forecast intervals is the historical errors method that supposes keeping constant an accuracy measure. The bootstrap method is also used when the distribution type of the sample is unknown.

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A grid bootstrap was used to compute the median without bias by (Gospodinov, 2002, p. 86), but the evolution of the events should be characterized by a high persistence. The main disadvantage is the high volume of computations. (Guan, 2003, p. 79).

The sieve bootstrap technique allows for consistent estimators of conditional repartition in the case of non-parametric prediction intervals (Alonso, Pena and Romo, 2003, p. 182). In Romania there is a strong correlation between inflation and money, making us to believe if the money earning went too far (Croitoru, 2013, p. 6). Therefore, the inflation forecasting should be taken under control.

In this paper we used as forecasting methods to build prediction intervals the historical errors method and the bootstrap technique.

It would be necessary to continue the research and build some Bayesian forecast intervals and to compare the results with those obtained using usual methods. The paper continues with the methodological framework, providing different types of quarterly and annual forecast intervals for the two variables. Moreover, the intervals are constructed using as point forecasts the anticipations of two forecasters during 2004-2015. It seems that first expert generated better inflation forecast intervals than the second one.

2. Methodology

The prediction interval that uses the historical errors method considers that errors follow a normal distribution of zero mean and standard deviation that equals the root mean squared error (RMSE) of the histrorical forecasts. Given a certain level of significance (α), the forecast intervals are built as it follows:

$$[\hat{y}_f - RMSE(k) \cdot z_{\left(\frac{\alpha}{2}\right)}; \hat{y}_f + RMSE(k) \cdot z_{\left(\frac{\alpha}{2}\right)}] \qquad (1)$$

$\hat{y}_f -$ the point prediction of the variable Y given at time t for the period (t+k)

$z_{\left(\frac{\alpha}{2}\right)}$- quantile α/2 of normal distribution of zero mean and standard deviation equalled to1

The following multiple linear regression is considered:

$$Y = X\beta + u \qquad (1)$$

Y- vector (length: nx1)

X- matrix (length: nxp)

β- vector of parameters (length: px1)

u- vector of error terms (length: nx1)

$\hat{u}$- residuals ($\hat{u}$=Y-X$\hat{\beta}$)

$\hat{\beta}$- parameter estimator ($\hat{\beta} = (X^T X)^{-1} X^T Y$)

The form of bootstrap model is:

$$Y^* = X\hat{\beta} + u^* \quad (2)$$

Y*- vector (length: nx1)

X*- matrix (lenth: nxp)

$\hat{\beta}$- parameter estimator ($\hat{\beta} = (X^T X)^{-1} X^T Y$)

u^* −random element

The selected sample is: $\{y_i^*\}_{i=1}^n$. The random term from theoretical bootstrap process uses modified residuals:

$$\tilde{u}_i = \frac{\hat{u}_i}{\sqrt{1-h_i}} - \frac{1}{n}\sum_{s=1}^{n} \frac{\hat{u}_s}{\sqrt{1-h_s}} \quad (3)$$

The theoretical process is computed as:

$$y_i^*(b) = X_i\hat{\beta} + \tilde{u}_i^*(b) \quad (4)$$

i=1,2,..,n

b-order of iteration

$\tilde{u}_i^*(b)$- resampled from $\tilde{u}_i$

Given the random variable z_j ($z_j = \frac{\hat{\beta}_j - \beta_j}{s(\hat{\beta}_j)}$), the interval for β_j considers that z_j has Student distribution (n-p degrees of freedom). For a level of confidence of (1-2α) the interval is:

$$[\ddot{\beta}_j - s(\ddot{\beta}_j)t_{(1-\alpha),n-p}; \ddot{\beta}_j - s(\ddot{\beta}_j)t_{(\alpha),n-p}] \quad (5)$$

$$[\hat{\beta}_j^*(\alpha B); \hat{\beta}_j^*((1-\alpha)B)] \quad (6)$$

The percentile-t bootstrap method is based on $z_j$estimation. We build a bootstrap table, the values of z_j^* are:

$$z_j^* = \frac{\beta_j^* - \beta_j}{s^*(\hat{\beta}_j^*)} \tag{7}$$

The percentile-t forecast interval for β_j is:

$$[\ddot{\beta}_j - s(\ddot{\beta}_j)\hat{t}^{(1-\alpha)}; \ddot{\beta}_j - s(\ddot{\beta}_j)\hat{t}^{(\alpha)}] \tag{8}$$

For the observation with number f of the exogenous variable X, the prediction is calculated using the model (Y- dependent variable): $\hat{y}_f = X_f \ddot{\beta}$. Having a normal distribution of the errors and the confidence interval (1-2α), the standard prediction interval is:

$$[\hat{y}_f - s_f \cdot t_{(1-\alpha),n-p}; \hat{y}_f + s_f \cdot t_{(1-\alpha),n-p}] \tag{9}$$

A prediction interval for y_f is based on forecast error $e_f = \hat{y}_f - y_f$. The future value y_f^*:

$$y_f^* = X_f \hat{\beta} + \tilde{u}_f^* \tag{10}$$

It is based on a retrieval of an empirical distribution of the modified residuals. For replication b, the prediction error is:

$$\hat{y}_f^*(b) = X_f \hat{\beta}^*(b) \tag{11}$$

$$e_f^*(b) = \hat{y}_f^*(b) - y_f^*(b) \tag{12}$$

The bootstrap forecast error is:

$$e_f^* = \hat{y}_f^* - \hat{y}_f - \tilde{u}_f^* \tag{13}$$

Given the empirical distribution of e_f^* (G^*), the percentiles are employed to determine the bootstrap prediction intervals $(G^{*-1}(1-\alpha) and\, G^{*-1}(\alpha))$. The percentile prediction interval is:

$$[\hat{y}_f - G^{*-1}(1-\alpha); \hat{y}_f - G^{*-1}(\alpha)] \tag{14}$$

For percentile-t prediction interval, standard deviation estimator (s^*) is :

$$s_f^* = s^* \cdot \sqrt{(1 + h_f} \tag{15}$$

$$h_f = X_f (X^T X)^{-1} X_f^T \tag{16}$$

The statistic z_f^* is determined:

$$z_f^* = \frac{e_f^*}{s_f^*} = \frac{\hat{y}_f^* - \hat{y}_f - \hat{u}_f^*}{s_f^*} \quad (17)$$

The percentile-t prediction interval has the form:

$$[\hat{y}_f - s_f \cdot z_{f(1-\alpha)}^*; \hat{y}_f - s_f \cdot z_{f(\alpha)}^*] \quad (18)$$

3. Forecast Intervals

Using the quarter point forecasts and the prediction intervals provided by the National Bank of Romania, we built some forecast intervals based on historical errors methods by keeping constant the root mean square error (RMSE) of the previous 4 quarter. The horizon is 2011:Q1-2015:Q4.

Table 1. Forecast intervals for the inflation rate predicted by National Bank of Romania (2011:Q1-2015:Q4)

Quarter	Forecast interval		Forecast historical method	interval-error	Point forecast	Actual values
	Lower limit	Upper limit	Lower limit	Upper limit		
T1:2011	7.48	7.95	**-2.96**	**17.96**	7.5	1.013
T2:2011	7.93	8.05	**-0.73**	**16.59**	7.93	1.005
T3:2011	3.45	3.58	**-1.69**	**8.59**	3.45	0.990
T4:2011	3.14	3.25	**-0.99**	**7.27**	3.14	1.012
T1:2012	1.43	2.52	**-3.01**	**7.81**	2.40	1.010
T2:2012	1.35	3.44	**-5.02**	**9.10**	2.04	1.002
T3:2012	2.46	5.20	**-3.40**	**14.06**	5.33	1.022
T4:2012	1.57	4.93	**-4.42**	**14.32**	4.95	1.009
T1:2013	1.34	5.33	**-3.78**	**14.16**	5.19	1.003
T2:2013	1.04	5.98	**-2.91**	**14.69**	5.89	1.003
T3:2013	**0.62**	**4.77**	**-4.04**	**11.06**	3.51	0.988
T4:2013	**0.81**	**5.18**	**-2.18**	**9.20**	3.51	1.006
T1:2014	1.02	7.93	-1.24	6.60	2.68	
T2:2014	1.5	3.5	-1.14	6.70	2.78	
T3:2014	1.5	3.5	-0.84	7.00	3.08	
T4:2014	1.5	3.5	-0.73	7.11	3.19	
T1:2015	1.5	3.5	-1.72	6.12	2.2	
T2:2015	1.5	3.5	-2.12	5.72	1.8	

T3:2015	1.5	3.5	-1.32	6.52	2.6
T4:2015	1.5	3.5	-1.12	6.72	2.8

Source: own computations

In the period from 2011 to 2013 only two forecast intervals of NBR include the actual values of inflation rate. The prediction intervals based on historical RMSE contain all the actual values during 2011-2013.

The variables with quarterly data that are used are: index of consumer prices that will be used in computing inflation rate, real exchange rate and unemployment rate on the period 2000:Q-2014:Q4. The quarterly forecasts will be made for 2011-2014, after the aggregation of data for obtaining annual values. The Tramo/Seats method was applied to get seasonally adjusted data. The logarithm was applied for the index of consumer prices. The data in first difference was computed for unemployment rate and exchange rate (d_ur and d_er).

The seasonally adjusted and stationarized index of consumer prices is denoted by log_ip. The following valid model was obtained:

$\log\_ip_t = 0{,}119 - 0{,}026 \cdot d\_er$ (19)

(std. error=0,08) (std. error=0,02)

(t-calc.=13.62) (t-calc.=-11.24)

According to Breusch-Godfrey test for the first lag, the errors are independent. The hypothesis of errors normal distrbution is checked using Jarque-Bera test and we do not have enought evidence to reject the normal repartition. According to White test, he errors are homoskedastic. The results of the application of these test are presented in Appendix 1.

For the seasonally adjusted and first differentiated quarterly unemployment rate (ur) an autoregressive model of order 1 is built, for which the errors are independent, homoskedastic and they follow a normal repartition (Appendix 2).

$\Delta ur_t = 0{,}005 + 0{,}309 \cdot \Delta ur_{t-1}$ (20)

Table 2. Point forecasts and bootstraped forecast intervals using the linear regression model for inflation rate (%) (percentile-t method) (horizon: 2011:Q1-2015:Q4)

Quarter	Point forecasts	Forecast intervals for inflation rate Intervals limits		Actual values
Q1:2011	1.0114	**0.0245**	**1.9983**	1.013
Q2:2011	1.0088	**0.0219**	**1.9957**	1.005
Q3:2011	1.0059	**0.0190**	**1.9928**	0.990

Q4:2011	1.0075	**0.0206**	**1.9944**	1.012
Q1:2012	1.0044	**0.0175**	**1.9913**	1.010
Q2:2012	1.0032	**0.0163**	**1.9901**	1.002
Q3:2012	1.0012	**0.0143**	**1.9881**	1.022
Q4:2012	1.0047	**0.0178**	**1.9916**	1.009
Q1:2013	1.0035	**0.0166**	**1.9904**	1.003
Q2:2013	1.0029	**0.0160**	**1.9898**	1.003
Q3:2013	1.0031	**0.0162**	**1.9900**	0.988
Q4:2013	1.0032	**0.0163**	**1.9901**	1.006
Q3:2014	1.0034	0.0165	1.9903	
Q4:2014	1.0033	0.0164	1.9902	
Q3:2014	1.002	0.0151	1.9889	
Q4:2014	1.0021	0.0152	1.9890	
Q3:2015	1.002	0.0151	1.9889	
Q4:2015	1.0013	0.0144	1.9882	
Q3:2015	1.0012	0.0143	1.9881	
Q4:2015	1.001	0.0141	1.9879	

Source: authors' computations

As we can see in the table above, the inferior and superior limits of the bootstrap intrvals have ranges with low variations. The results are close of the desired monetary policy in Romania, but the intervals are too narrow and the registered inflation rate for inflation is located out of these intervals. The reasons for this fact are related to the underestimated point forecasts for inflation based on linear regression model. All the forecast intervala based on percentile-t method include the actual values of inflation rate.

Table 3. Point forecasts and bootstraped forecast intervals using the linear regression model for unemployment rate (%) (percentile-t method) (horizon: 2011:Q1-2015:Q4)

Quarter	**Point forecasts**	**Forecast intervals for inflation rate Intervals limits**		**Actual values**
Q1:2011	7.21	**5.46**	**8.95**	7.20
Q2:2011	7.27	**5.52**	**9.01**	7.40
Q3:2011	7.41	**5.66**	**9.15**	7.40
Q4:2011	7.41	**5.66**	**9.15**	7.40
Q1:2012	7.37	**5.63**	**9.12**	7.30
Q2:2012	7.21	**5.47**	**8.96**	7.00
Q3:2012	7.04	**5.29**	**8.78**	7.10
Q4:2012	7.07	**5.33**	**8.82**	7.00
Q1:2013	7.07	**5.32**	**8.81**	7.20
Q2:2013	7.24	**5.49**	**8.98**	7.30
Q3:2013	7.31	**5.56**	**9.05**	7.30
Q4:2013	7.31	**5.56**	**9.05**	7.30

Q1:2014	7.33	**5.59**	**9.07**	7.20
Q2:2014	7.4	**5.66**	**9.14**	7.20
Q3:2014	7.41	5.67	9.15	
Q4:2014	7.43	5.69	9.17	
Q1:2015	7.45	5.71	9.19	
Q2:2015	7.45	5.71	9.19	
Q3:2015	7.5	5.76	9.24	
Q4:2015	7.53	5.79	9.27	

Source: authors' computations

Starting with 2013, the unemployment rate has a slow tendency of increase. The variations of range for forecast intervals for unemployment rate are rather small, because the differencies between predicted unemployment are low from a quarter to another. All the forecast intervals based on percentile-t method include the actual values of unemployment rate.

Table 4. Point forecasts and forecast intervals for qurterly inflation rate and unemployment rate (%) based on historical error methods (horizon: 2011:Q1-2015:Q4)

Quarter	Forecast intervals of inflation rate based on historical RMSE of the previous 4 quarters		Forecast intervals of unemployment rate based on historical RMSE of the previous 4 quarters	
	Intervals limits		Intervals limits	
Q1:2011	-9.448	11.471	6.86	7.55
Q2:2011	-7.655	9.673	6.93	7.60
Q3:2011	-4.130	6.142	7.03	7.78
Q4:2011	-3.121	5.136	6.92	7.89
Q1:2012	-4.407	6.415	6.83	7.91
Q2:2012	-6.057	8.063	6.65	7.77
Q3:2012	-7.731	9.734	6.46	7.61
Q4:2012	-8.365	10.375	6.58	7.57
Q1:2013	-7.963	9.970	6.58	7.55
Q2:2013	-7.799	9.805	6.79	7.68
Q3:2013	-6.551	8.557	6.91	7.70
Q4:2013	-4.687	6.694	6.95	7.66
Q1:2014	-2.917	4.923	7.10	7.56
Q2:2014	-2.917	4.923	7.09	7.71
Q3:2014	-2.918	4.922	7.03	7.79
Q4:2014	-2.919	4.920	7.00	7.86
Q1:2015	-2.921	4.917	6.96	7.94
Q2:2015	-2.923	4.915	7.05	7.67
Q3:2015	-2.928	4.912	7.34	7.87
Q4:2015	-2.929	4.911	7.56	7.96

Source: authors' computations

Forecasts of inflation and unemployment rate provided by this method seem reasonable,the lenght of intervals being rather big. However, if we go in time, these intervals become narrower. All the forecast intervala based on historical error method include the actual values of inflation and unemployment rate.

Considering constant the error from previous year, we will build forecast intervals for inflation and unemployment rate based on two experts' predictions on the horizon 2004-2015. Some point forecasts are provided by (Dobrescu, 2013, p. 10).

Table 5. Prediction intervals for annual inflation rate (%) based on historical errors method (horizon: 2004-2015)

Year	Forecast intervals based on first expert forecasts		Forecast intervals based on second expert predictions		Actual inflation rate
2004	**3.99**	**18.97**	**5.24**	**18.56**	15.3
2005	**10.13**	**17.35**	**3.32**	**14.68**	11.9
2006	**7.82**	**9.38**	**3.08**	**10.92**	9
2007	**3.90**	**7.42**	**9.4**	**8.06**	6.56
2008	**1.33**	**15.67**	6.03	11.7	4.84
2009	**1.19**	**10.01**	7.93	11.07	7.85
2010	**4.81**	**7.99**	**5.00**	**7.40**	5.59
2011	**3.17**	**7.04**	8.29	9.931	6.09
2012	**2.15**	**6.85**	5.37	10.263	3.3
2013	**-3.19**	**12.93**	-4.58	2.13	3.98
2014	-3.194	4.806	-07.18	2.10	
2015	-3.628	5.638	-8.18	2.2201	

Source: authors' computations

The intervals range for inflation rate is extremly variable in the period 2004-2012. The range is larger during 2013-2015. All the forecast intervals based on first expert anticipations include the actual values of inflation rate while only 5 out of 10 intervals on the horizon 2004-2013 contain the second expert prognosis.

Table 6.Forecast intervals for annual unemployment rate (%) based on historical errors method (horizon: 2004-2015)

Year	Forecast intervals based on first expert forecasts		Forecast intervals based on second expert predictions		Actual unemployment rate
2004	**6.808**	**7.592**	**6.8240**	**9.1760**	7.4
2005	**4.754**	**11.066**	**4.7640**	**11.0360**	6.3
2006	**4.748**	**9.452**	**4.0760**	**11.5240**	5.9
2007	**1.638**	**11.282**	**0.5440**	**14.6560**	4
2008	**3.536**	**7.064**	**1.5200**	**13.2800**	4.4
2009	**3.400**	**13.200**	**3.3040**	**13.4960**	5.8

2010	**6.636**	**10.164**	**7.2040**	**7.5960**	7.5
2011	**6.604**	**7.812**	**6.3240**	**8.6760**	6.9
2012	**4.748**	**9.452**	**4.2680**	**10.9320**	5.9
2013	3.136	6.664	3.4320	6.5680	7.3
2014	5.836	9.364	5.4320	8.5680	
2015	5.945	9.567	5.4734	8.5834	

Source: authors' computations

In 2007 the highest range for prediction intervals was obtained for both experts. 9 out of 10 forecast intervals based on first expert anticipations and the second one predictions include the actual values of inflation rate during 2004-2013. For the last year in the horizon both forecasters anticipated lower unemployment rates.

4. Conclusion

The forecast intervals are a way to reflect the uncertainty that affects the forecasting process. For inflation rate and unemployment rate point predictions forecast intervals were built for Romania, providing a better framework for establishing the decision making process. The annual inflation rate forecasts of the first expert anticipation generated precise prediction intervals when bootstrapping and historical errors methods are applied during 2004-2013. However, the intervals are quite large. A future direction of research would be the construction of forecast intervals using Bayesian method.

5. Acknowledgement

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6. References

Alonso, M., Pena, D. & Romo, J. (2000). Sieve Bootstrap Prediction Intervals. *Proceedings in Computational Statistics 14th Symposium*, pp. 181-186, Utrecht.

Bratu, M. (2012). Forecast Intervals for Inflation in Romania. *Timisoara Journal of Economics*, *5*(1 (17)), pp. 145-152.

Croitoru, L. (2013). What Good is Higher Inflation? To Avoid or Escape the Liquidity Trap. *Romanian Journal of Economic Forecast*, Vol. 16, No. 3, pp. 5-25.

Dobrescu, E. (2013). Updating the Romanian Economic Macromodel. *Journal for Economic Forecasting*, Vol. 16, No. 4, pp. 5-31.

Gospodinov, N. (2002). Median unbiased forecasts for highly persistent autoregressive processes. *Journal of Econometrics*, Vol. 111, No. 1, pp. 85-101.

Guan, W. (2003). From the help desk: bootstrapped standard errors. *The Stata Journal*, Vol. 3, No. 1, pp. 71–80.

APPENDIX 1. Linear regression model for quarterly index of consumer prices

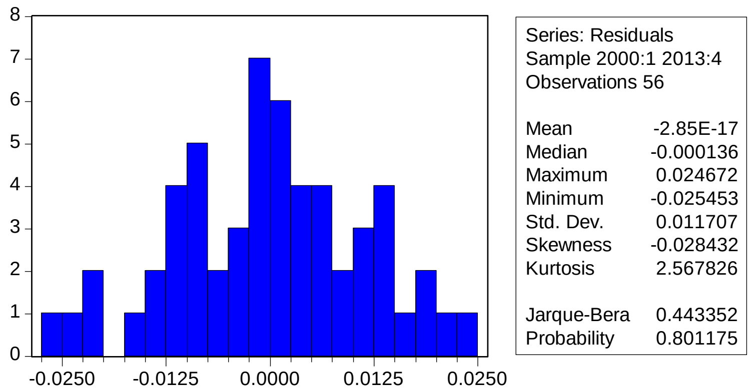
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.119341	0.008761	13.62264	0.0000
Curs_schimb__SA	-0.026202	0.002331	-11.24136	0.0000
R-squared	0.700613	Mean dependent var		0.022474
Adjusted R-squared	0.695068	S.D. dependent var		0.021395
S.E. of regression	0.011814	Akaike info criterion		-6.003922
Sum squared resid	0.007537	Schwarz criterion		-5.931588
Log likelihood	170.1098	F-statistic		126.3683
Durbin-Watson stat	1.032398	Prob(F-statistic)		0.000000

White Heteroskedasticity Test:

F-statistic	1.284795	Probability	0.285184
Obs*R-squared	2.589492	Probability	0.273967

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	6.08290	Probability	0.191
Obs*R-squared	3.03713	Probability	0.305



APPENDIX 2. Autoregressive model for quarterly unemployment rate

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.005242	0.042195	0.124224	0.9016
U(-1)	0.309934	0.131379	2.359076	0.0221

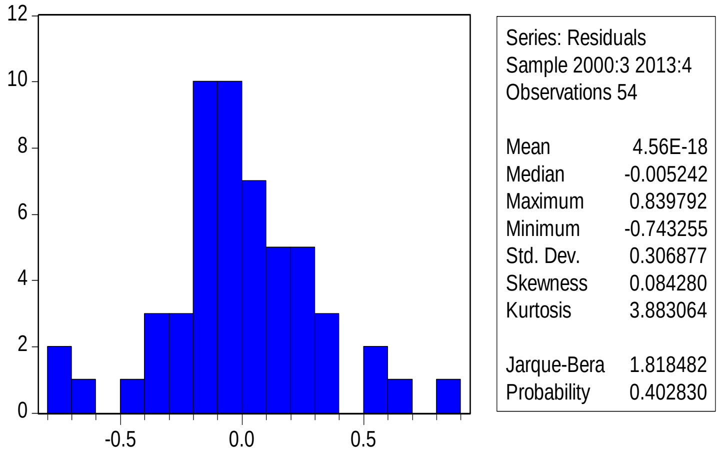
R-squared	0.096677	Mean dependent var	0.009259
Adjusted R-squared	0.079305	S.D. dependent var	0.322881
S.E. of regression	0.309814	Akaike info criterion	0.530642
Sum squared resid	4.991194	Schwarz criterion	0.604308
Log likelihood	-12.32734	F-statistic	5.565238
Durbin-Watson stat	1.894308	Prob(F-statistic)	0.022112

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	1.422747	Probability	0.238472
Obs*R-squared	1.465554	Probability	0.226049

White Heteroskedasticity Test:

F-statistic	0.352616	Probability	0.704547
Obs*R-squared	0.736532	Probability	0.691933


12
10
8
6
4
2
0
-0.5
0.0
0.5
Series: Residuals
Sample 2000:3 2013:4
Observations 54
Mean 4.56E-18
Median -0.005242
Maximum 0.839792
Minimum -0.743255
Std. Dev. 0.306877
Skewness 0.084280
Kurtosis 3.883064
Jarque-Bera 1.818482
Probability 0.402830

A Comparative Analysis of Some Results from $\mathbf{Q}_p$ and R

Alin Cristian Ioan[1]

Abstract: The paper investigates whether a series of concepts and properties available in the real analysis remains valid for p-adic case. There are many similarities between **R** and $\mathbf{Q}_p$ and also so many differences. First of all, **R** is an ordered field, which is not true for $\mathbf{Q}_p$. Secondly **R** is archimedean (that is the absolute valuation $|\bullet|$ is archimedean) while $\mathbf{Q}_p$ is not archimedean for any p prime. This means that **R** is a connected metric space while $\mathbf{Q}_p$ is totally disconnected. This proves that there is no analogous notion of interval in $\mathbf{Q}_p$ or a notion similar to the curve. These contrasts will cause the difference between the analysis p-adic and the real analysis.

Keywords: p-adic; sequences; series; function

JEL Classification: C02

1 Introduction

Let note $\mathbf{Q}_p$ the field of p-adic numbers. Before we begin, we should note that there are many similarities between **R** and $\mathbf{Q}_p$ and also so many differences. First of all, **R** is an ordered field, which is not true for $\mathbf{Q}_p$. Secondly **R** is archimedean (that is the absolute valuation $|\bullet|$ is archimedean) while $\mathbf{Q}_p$ is not archimedean for any p prime. This means that **R** is a connected metric space while $\mathbf{Q}_p$ is totally disconnected. This proves that there is no analogous notion of interval in $\mathbf{Q}_p$ or a notion similar to the curve. These contrasts will cause the difference between the analysis p-adic and the real analysis.

2 Sequences and Series in $\mathbf{Q}_p$

We begin by studying the basic properties of strings and series in $\mathbf{Q}_p$. The most important thing about $\mathbf{Q}_p$ is that the field is a complete field, therefore every

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Cauchy sequence is convergent. Naturally all the properties of the norm $|\cdot|_\infty$ on **R** are the same of the properties of the p-adic valuations (the property of being non-archimedean being an additional property).

As a result, many of the basic theorems that occur in the real analysis, taking place also in the p-adic analysis. One of the great benefits of the p-adic analysis is that it will bring generalizations to some real questions raised in the analysis (due to the property of $|\cdot|_p$ to be non-archimedean).

Lemma 1

A sequence $(x_n)\subset\mathbf{Q}_p$ is a Cauchy sequence if and only if $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$.

Proof

If m=n+r>n, we get $|x_m - x_n| = |x_{n+r} - x_{n+r-1} + x_{n+r-1} - x_{n+r-2} + \ldots - x_n| \leq \max \{ |x_{n+r} - x_{n+r-1}|, |x_{n+r-1} - x_{n+r-2}|, \ldots, |x_{n+1} - x_n| \}$ this fact being true because $|\cdot|_p$ is non-archimedean. Now for $\forall\, r \in \mathbf{N}^*$ and $\varepsilon > 0\ \exists\, N_\varepsilon \in \mathbf{N}^*$ such that $|x_m - x_n| = |x_{n+r} - x_n| \leq \max \{ |x_{n+r} - x_{n+r-1}|, |x_{n+r-1} - x_{n+r-2}|, \ldots, |x_{n+1} - x_n| \} < \varepsilon\ \forall\, n, m \geq N_\varepsilon$. N_ε is that natural number which $\forall\, n \geq N_\varepsilon$ we have $|x_{n+1} - x_n| < \varepsilon$. Therefore, the sequence $(x_n) \subset \mathbf{Q}_p$ is Cauchy so convergent.♦

The theory of sequences and their convergence is therefore similar with that on **R** except lemma above.

Proposition 2

Let $(a_n) \subset \mathbf{Q}_p$ a convergent sequence. Then we have one of two statements: either $\lim |a_n| = 0$, or there exists an integer M such that $|a_n| = |a_M|\ \forall\, n \geq M$. In other words, the absolute value of the sequence converges to zero or it becomes constant after a rank on.

Proof

Suppose that $\lim |a_n| \neq 0 \Rightarrow \exists\, \varepsilon > 0$ such that $\forall\, N_1 \in \mathbf{N}^*$, $\exists\, n \geq N_1$ with $|a_n| > \varepsilon$. So $\exists$ a number $c > \varepsilon > 0$ with $|a_n| \geq c > \varepsilon$, $\forall\, n \geq N_1$. On the other hand $\exists\, N_2$ integer for which $\forall\, n,m \geq N_2 \Rightarrow |a_n - a_m| < c$. We want both conditions occur so fix $\forall\, \varepsilon > 0$ $N = \max\{N_1, N_2\}$. Now $\forall\, n,m \geq N \Rightarrow |a_n - a_m| < \max\{|a_n|, |a_m|\}$ from where we get $|a_n| = |a_m|$ after non-archimedean property (that is, in the space $\mathbf{Q}_p$ all triangles are isosceles).♦

Also, for series the classical theory remains valid. For example, the following statements are true:

Proposition 3

Let $(a_n) \subset \mathbf{Q}_p$. The absolute convergence of sequence implies its convergence, ie if a series of absolute values $\sum |a_n|$ converges in **R** then the series $\sum a_n$ converges in $\mathbf{Q}_p$.

Proof

The series $\sum a_n$ converges in $\mathbf{Q}_p \Leftrightarrow \lim |a_n| = 0$. But a necessary condition for absolute series to converges is that $\lim |a_n| = 0$.♦

The next result is a strong result in real analysis, but in p-adic context, the previous lemma becomes an important tool to determine whether a series of p-adic numbers converges in $\mathbf{Q}_p$ namely:

Corollary 4

An infinite series $\sum_{n=0}^{\infty} a_n$ with $(a_n) \subset \mathbf{Q}_p$ is convergent $\Leftrightarrow \lim_{n\to\infty} a_n = 0$. In this case we also have $|\sum_{n=0}^{\infty} a_n| \leq \max_n |a_n|$.

Proof

A series converges only when the sequence of partial sums converges. Now take the difference between the n-th partial sum and the (n-1)-th. By Lemma we get that this difference tends to 0 as we wanted. Conversely we have the sequence of partial sums is Cauchy therefore convergent. If $\sum_{n=0}^{\infty} a_n=0$ we have nothing to prove. Otherwise, for any partial sum, we have $|\sum_{n=0}^{N} a_n| \leq \max_{0\leq n\leq N} |a_n|$. Since $\lim_{n\to\infty} a_n = 0 \Rightarrow \forall\ \varepsilon > 0\ \exists\ N_\varepsilon \in \mathbf{N}^*$ such that $|a_n| < \varepsilon\ \forall\ n > N_\varepsilon = N$. Let $\varepsilon = \max_{0\leq n\leq N} |a_n|$. Thus we have $\max_{0\leq n\leq N} |a_n| = \max_n |a_n|$. How $\max_n |a_n|$ does not depend on N for $N \to \infty$ we get $|\sum_{n=0}^{\infty} a_n| \leq \max_n |a_n|$, that is the conclusion.♦

The reciprocal question related to when a series is convergent in **R** implies that its general term tends to zero is not necessarily true. As a counterexample we have the harmonic series which not converges in **R**.

Therefore, it is much easier to establish convergence of the infinite series in p-adic context than in **R**. This seems to express that the theory of series in $\mathbf{Q}_p$ is much simpler than in **R**.

Now we shall consider a "double string" $(b_{ij}) \subset \mathbf{Q}_p$ asking what happens to the two series considered after a summing with i and after j or viceversa. For this, it is necessary that, as example, $b_{ij} \to 0$ when one of the indices is fixed and the other goes to infinity (otherwise obvious series will not converges). We shall say that

$\lim_{i\to\infty} b_{ij}$ = 0 uniformly in j if $\forall$ ε> 0 we can find an integer N which does not depend on j such that $\forall i \geq N \ \forall \Rightarrow |\ b_{ij}\ | < \varepsilon \ \forall$ j. In other words, the sequence (b_{ij}) tends to 0 when $i \to \infty$, the convergence coming from the same rank for all j. First we prove the following lemma:

Lemma 5

Let $(b_{ij}) \subset \mathbf{Q}_p$ and assume that:

1) $\forall$ i, $\lim\limits_{j\to\infty} b_{ij} = 0$

2) $\lim\limits_{i\to\infty} b_{ij} = 0$ uniformly in j

Then for any real number ε > 0 ∃ an integer N_ε which depends only of ε such that if $\max(i, j) \geq N \Rightarrow |b_{ij}| < \varepsilon$.

Proof

Let ε > 0 fixed. The second condition says that we can choose $N_0 \in \mathbf{N}^*$, which depends on ε but not of j such that $|b_{ij}| < \varepsilon$ if $i \geq N_0$. The first condition is weaker (it says basically that $\forall$ i we can find $N_1(i\ ) \in \mathbf{N}^*$, "the notation suggesting that the whole depends on i") such that if $j \geq N_1(i\ )$ we have $|b_{ij}| < \varepsilon$. Now we take $N = N(\varepsilon) = \max(N_0, N_1(0), N_1(1), \ldots, N_1(N_0 - 1))$. The choice of N was done so that if $\max(i,j\ ) \geq N$ then $i \geq N_0$ when $|b_{ij}| < \varepsilon$ regardless of j or if $i < N_0 \Rightarrow j \geq N$ and $i \in \{0,1,2,\ldots, N_0 - 1\}$ therefore $j \geq N_1(i\ )$, when we have $|b_{ij}| < \varepsilon$.♦

Proposition 6

Let $(b_{ij}) \subset \mathbf{Q}_p$ and assume that:

1) $\forall$ i, $\lim\limits_{j\to\infty} b_{ij} = 0$

2) $\lim\limits_{i\to\infty} b_{ij} = 0$ uniformly in j

Then the series $\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} b_{ij})$ and $\sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} b_{ij})$ converges and their sums are equal.

Proof

From the previous lemma we know that for a given ε > 0 we can choose N such that for $\max(i, j) \geq N \Rightarrow |b_{ij}| < \varepsilon$. In particular for $\forall$ i, when $j \to \infty$ or viceversa then the inner sums $\sum_{j=0}^{\infty} b_{ij}$ and $\sum_{i=0}^{\infty} b_{ij}$ converges (the first sum for each i and the second for each j). More, for $i \geq N$ we have $|\sum_{j=0}^{\infty} b_{ij}| \leq \max_j |b_{ij}| < \varepsilon$.

Similarly for any j ≥ N we have | $\sum_{i=0}^{\infty} b_{ij}$| < ε. In particular, we note that $\lim_{i\to\infty} \sum_{j=0}^{\infty} b_{ij}$ = 0 and $\lim_{j\to\infty} \sum_{i=0}^{\infty} b_{ij}$ = 0 therefore both series converges. It remains to show that the sums of the two double series are equal. We will continue to use N and ε as above so that the condition $|b_{ij}|$ < ε ∀ i or j ≥ N holds. We will often use the ultrametric inequality: |x + y| ≤ max{ |x|, |y|} applied even at the level of series as we have seen in the last corollary. We see first | $\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} b_{ij}) - \sum_{i=0}^{N}(\sum_{j=0}^{N} b_{ij})$| = | $\sum_{i=0}^{N}(\sum_{j=N+1}^{\infty} b_{ij}) - \sum_{i=N+1}^{\infty}(\sum_{j=0}^{\infty} b_{ij})$ |. Now for j ≥ N+1 we shall have $|b_{ij}|$ < ε ∀ i. With ultrametric inequality it remains that | $\sum_{j=N+1}^{\infty} b_{ij}$| < ε ∀ i and, using again the ultrametric inequality we have that | $\sum_{i=0}^{N}(\sum_{j=N+1}^{\infty} b_{ij})$| < ε. Similarly, we obtain |$\sum_{i=N+1}^{\infty}(\sum_{j=0}^{\infty} b_{ij})$ | < ε. So, again applying this inequality we have that: | $\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} b_{ij}) - \sum_{i=0}^{N}(\sum_{j=0}^{N} b_{ij})$| < ε. Reversing now i with j we get a similar inequality that is | $\sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} b_{ij}) - \sum_{j=0}^{N}(\sum_{i=0}^{N} b_{ij})$| < ε, then finally | $\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} b_{ij}) - \sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} b_{ij})$| < ε. But how ε was arbitrarily fixed the double series are equal.♦

What basically says this proposition is that if the double sequence {bij} converges to 0 in a uniform way, then the double sum after i and j can be taken in any order to give the same answer.

Now if a = $\sum a_n$ and b = $\sum b_n$ are two convergent series, then the series $\sum a_n$ + b_n is convergent and has the sum a + b. Indeed, the first sum is convergent ⟺ $\lim_{n\to\infty} a_n$ = 0 and so the second if $\lim_{n\to\infty} b_n$ = 0. In conclusion, $\lim_{n\to\infty} a_n + b_n$ = 0, which is enough to say that the series $\sum a_n$ + b_n converges. Now, noting with c the sum of the series we have that $\sum_{k=0}^{n}(a_k + b_k)$ = $\sum_{k=0}^{n} a_k$ + $\sum_{k=0}^{n} b_k$ and passing to the limit with n → ∞ we have that c = a + b.

A second problem is related in some way to the top as follows: if a = $\sum a_n$ and b = $\sum b_n$ are two convergent series, taking $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ then the series $\sum c_n$ is convergent and its sum is ab.

Let the partial sum of order n of a and the partial sum of order n of b that is s_n = $\sum_{k=0}^{n} a_k$ and t_n = $\sum_{l=0}^{n} b_l$. Now $s_n t_n = \sum_{k=0}^{n} \sum_{l=0}^{n} a_k b_l$. As above, we have: $\lim_{n\to\infty} a_n$ = 0 and $\lim_{n\to\infty} b_n$ = 0. Computing $\sum_{k=0}^{n} a_k \cdot \sum_{l=0}^{n} b_l$ - $\sum_{l=0}^{n} \sum_{k=0}^{n} a_k b_l$ for n ∈ **N**. In short, this expression is written $s_n t_n$ - c_n - $\sum_{l+k\neq n} a_k b_l$ = 0 where l and k go through the set of numbers 0,...,n. Finally, we have: $s_n t_n - c_n - c_{n-1} - c_{n-2} + ... - c_0 - c_{n+1}$ $- ... - c_{2n} = 0$ ie passing to the limit with $n \to \infty$ we get ab = $\sum c_n$ that is c = ab.

3 Functions, Continuity, Differentiability in $\mathbf{Q}_p$

The basic idea on the functions and continuity remains unchanged by the passage of real numbers to p-adic numbers because ultimately they depend on the metric structure. Not be able to work with intervals (nay nor related with nontrivial connected sets), so that our functions will be defined on disks (closed-open). We shall write B(a,r) for open sets of center a and radius r> 0 and $\overline{B}$(a,r) for the closed sets of center and radius r.

Definition 7

Let $U \subset \mathbf{Q}_p$ be an open set. A function f: $U \to \mathbf{Q}_p$ is called continous in $a \in U$ if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in U$ with the property $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

The base results on continuity are true in all metric spaces and therefore also in the p-adic fields. For example, if U is a compact set (and remember that $\mathbf{Q}_p$ is both open and compact so a subset included in it can have these properties) and f is continuous at any point in U then f is uniformly continuous. Automaticaly, the Darboux property to carry an interval within an interval is true since the intervals in $\mathbf{Q}_p$ are identified with points. In the general context, the Darboux property says that a continuous function defined on a metric space carry a connected set into another connected set.

Now, if $U = \mathbf{Z}_p$ then for any $a \in \mathbf{Z}_p$, $\forall \varepsilon > 0$, $\exists n \in \mathbf{N}$ with $\forall x \in \mathbf{Z}_p$ such that $|x - a| < \frac{1}{p^n}$, we have $|f(x) - f(a)| < \varepsilon$. However $\varepsilon = \frac{1}{p^m}$ for $m \in \mathbf{Z}$. For m = 0 we have that $f(x) - f(a) \in \mathbf{Z}_p$ that is f(x) is in one of the neighbourhoods (closed-open) of f(a) ie f carry a local connected set into a local connected set.

Derivatives are perhaps more interesting from the fact that there is a lower analogy with the classical real case. It will make sense to define derivatives of functions f: $\mathbf{Q}_p \to \mathbf{Q}_p$ in the usual way, namely:

Definition 8

Let $U \subset \mathbf{Q}_p$ be an open set and let f: $U \to \mathbf{Q}_p$ a function. We say that f is differentiable in $x \in U$ if $\exists$ the limit $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. If $f'(x)$ exists for any $x \in U$ we shall say that f is differentiable on U and we write: $f^{'}: U \to \mathbf{Q}_p$ for the function $x \to f^{'}(x)$.

Remark

Up to a certain point, the derivative of a function with values in $\mathbf{Q}_p$ behaves as if real, that is it can be shown that a differentiable function is continuous as shown in **R** or **C**.

It is natural to ask what is the role of the derivative of a function in the p-adic case. But if we consider that the mean value theorem states for a and b real data in the domain of definition of a differentiable function (while continuing) $\exists\ \xi$ between a and b such that $f(b) - f(a) = f'(\xi)(b - a)$, is not working in the p-adic case, because in fact we have not the relation of "being between" because $\mathbf{Q}_p$ is not an ordered field. But this slight inconvenience can be simply remedied if we think that in **R** we can define the relation "being between" saying that ξ is between a and b if we have $\xi=at+b(1-t)$ for $0 \leq t \leq 1$. Nearly the same happens in the complex case. What we can now express through the mean value theorem in the p-adic case? We ask if there the statement holds: if we have a function f defined on $\mathbf{Q}_p$, differentiable and continuous on $\mathbf{Q}_p$ then for any two numbers a and b in $\mathbf{Q}_p$ $\exists\ \xi \in \mathbf{Q}_p$ of the form: $\xi=at+b(1-t)$ for t such that $|t| \leq 1$, for which $f(b) - f(a)=f^{'}(\xi)(b - a)$. We shall show that the mean value theorem for p-adic case is false.

Proof

Let $f(x) = x^p - x$, $a = 0$, $b = 1$. We have $f^{'}(x) = px^{p-1} - 1$ and $f(a) = f(b) = 0$. If the statement is true, it exists $\xi \in \mathbf{Q}_p$ of the form $\xi = at + b(1 - t) = 1 - t$ with $t \in \mathbf{Z}_p$ such that $p\xi^{p-1} - 1 = 0$. But from here and $\xi \in \mathbf{Z}_p$ and from $p\xi^{p-1} - 1 = 0 \Rightarrow 0 \in 1 + p\,\mathbf{Z}_p$ - contradiction.♦

4 References

Gouvea, Fernando Q. (1997). *P-adic numbers. An introduction*. 2nd Edition. New York, Heidelberg, Berlin: Springer-Verlag.

Ioan, A.C. (2013). *Through the maze of algebraic theory of numbers*. Galati: Zigotto Publishing.,

Ireland, Kenneth & Rosen Michael (1990). *A classical introduction to modern number theory*, Springer.

Marcus, Daniel (1977). *Number fields*. New York: Springer Verlag.

Neukirch, Jurgen (1999). *Algebraic number theory*. New York, Heidelberg, Berlin: Springer Verlag.

Roquette, Peter (2003). *History of valuation theory*. Part I, Heidelberg.