Miscellaneous

Moldova's Race Against time to Adhering to the European Union

Romeo-Victor Ionescu[1]

Abstract: The paper deals to the idea of supporting Moldova's adhering to the EU as soon as possible. In order to have a scientific approach of this process, the analysis is focused on four representative economic indicators (GDP growth rate, unemployment and inflation rates and public debt). The comparative analysis between EU average, Bulgaria, Romania and Moldova covers 2009-2015. It is coupled with the regression analysis, in order to quantify the disparities between the above four economic entities. The intermediate results of the analysis in the paper support the cluster approach for the four economies. A distinct part of the analysis is the use of forecasting procedures for all four economic indicators, in order to highlight the possibility of decreasing the economic disparities between Moldova and the other three economies at the end of 2020. The main conclusion of the paper is that Moldova is not able to achieve the EU average economic development even in 2020. This is why a political decision can be change this unoptimistic conclusion. The analysis in the paper is supported by the latest official statistic data, pertinent tables and diagrams.

Keywords: economic disparities; GDP growth rate; unemployment rate; inflation rate; public debt; economic forecasting.

JEL Classification: E60; F43; F63; O52; R11

1. General Approach

The political, military and socio-economic context at the EU's Easter borders became very dynamic. In order to stop the Russian expansion in the region, USA, Canada and U28 use economic sanctions. Moreover, EU28 supports Moldova and Ukraine in their future adhering process.

Moldova faces to the risk of being target for the Russian expansion. This is why the political elections' new context allows Moldova to choose for the European way.

The paper deals with the idea of quantifying Moldova's economic potential in order to forecast its adhering to the EU horizon. In order to realise this, the analysis is

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focused on the main macroeconomic indicators in Moldova, Romania, Bulgaria and EU average. Romania and Bulgaria were chosen as the lowest developed Member States and EU average is interesting in order to establish the economic disparities between Moldavian and European economies. On the other hand, the adhering decision becomes more a political than economic decision. This is why Moldova should have the opportunity to burn some steps on the adhering road.

2. Literature: Critical Overview

The post global crisis' economic recovery created new disparities across the Member States. This is why EU launched Europe 2020 Strategy, an ambitious project which should be support better the socio-economic cohesion at the end of 2020 (European Commission, 2010).

The economic environment's instability forced the European Commission to realize two forecasts every year. According to the latest official statistic data, EU28 will achieve better economic growth rate in 2015, but the employment and the inflation rates will stay high (European Commission, 2014, p. 1).

Bulgaria will achieve higher economic growth rate in 2015, but it will face to labour market's weakness and inflation during the same year (European Commission, 2014, p. 51).

On the other hand, Romania will succeed in labour market stabilisation and unemployment rate maintaining in 2015, but it will face with higher inflation (European Commission, 2014, p. 93).The GDP growth rate was 3.9% in the first semester of 2014 in Moldova (National Statistical Bureau of Moldova, 2014).

An interesting comparative analysis between Romania and Moldova was focused on the economic performances of both countries. The first conclusion of this analysis was that Romania and Moldova have the same situation as Eastern and Western Germany in 1990 and the GDP/capita rate is almost 4:1 (Anghel, I.& Cîrchelan, A., 2013).

On the other hand, Moldova started a powerful economic recovery process, in order to decrease the disparities to EU average. As a result, the GDP growth rate will increase constantly during 2014-2017, but they will not be able to achieve almost 9.0% as in 2013 (World Bank, 2014).

The latest economic forecast in Moldova talks about the need of making structural reforms in order to improve competitiveness and economic growth (Piontkivsky, R. & Chistruga, M., 2014).

3. Research Methodology

It is very difficult to quantify the macroeconomic evolution under a very volatile global economic environment.

The analysis in the paper used the most possible long statistic data related to the main macroeconomic indicators for EU28, Bulgaria, Romania and Moldova. As a result, GDP growth rate, unemployment rate, inflation rate and public debt are used in the analysis.

The comparative analysis is followed by regression, in order to highlight the disparities between the above four economic entities. The regression is applied using ANOVA conditions.

The analysis' results support the cluster approach in studying the macroeconomic performances of these countries. The two-step cluster analysis is realized under Euclidean distance measuring. Last but not the least, forecasting procedures are used in order to quantify the economic results at the end of 2020. The dependent variables are the annual values of the economic indicators, while the independent value is time. The forecasting method is ARIMA. All intermediate and final conclusions are supported by pertinent statistic tables and diagrams.

4. Economic Trends and Dynamics

The global crisis' impact was important not only across the less developed Member States, but on EU28 average and other European countries. The economic recovery started in 2010 is not yet finished in all Member States. As a result, the GDP growth rate presents high disparities (see Table 1).

Table 1. GDP growth rate (%)

	2009	2010	2011	2012	2013	2014	2015
EU28	0.9	2.0	1.7	-0.4	0.1	1.5	2.0
Bulgaria	2.9	0.4	1.8	0.8	0.6	1.7	2.0
Romania	3.2	-1.1	2.2	0.7	3.5	2.3	2.5
Moldova	-6.0	7.1	6.8	-0.8	8.9	2.0	3.0

According to Figure 2, Moldova will achieve highest GDP growth rates in 2014 and 2015, comparing to the other analyzed three economic entities.

Figure 1. GDP growth rate (%)

The GDP growth rate in 2014 supports the idea of cluster approach for the analysed economic entities (see Figure 2).

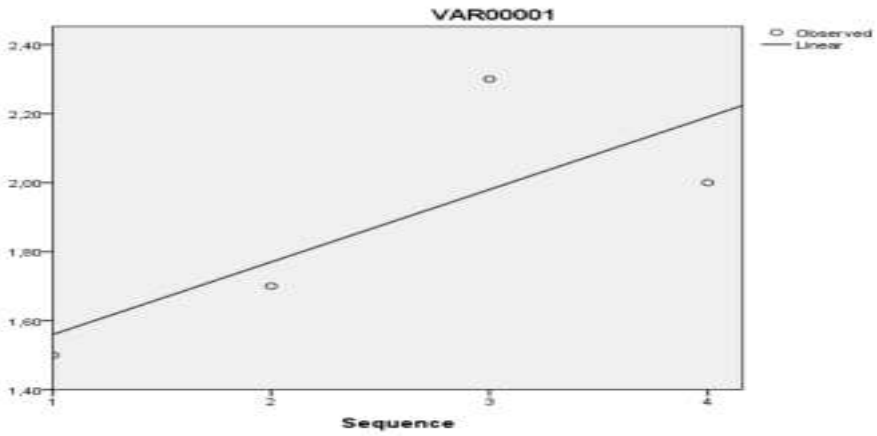


Figure 2. GDP growth rate's disparities

Source: personal contribution using IBM-SPSS software

Figure 2 allows introducing the cluster approach in the analysis. The paper uses two clusters: EU average and Bulgaria as the first cluster, and Romania and Moldova as the second one. The viability of this approach is presented in Figure 3.

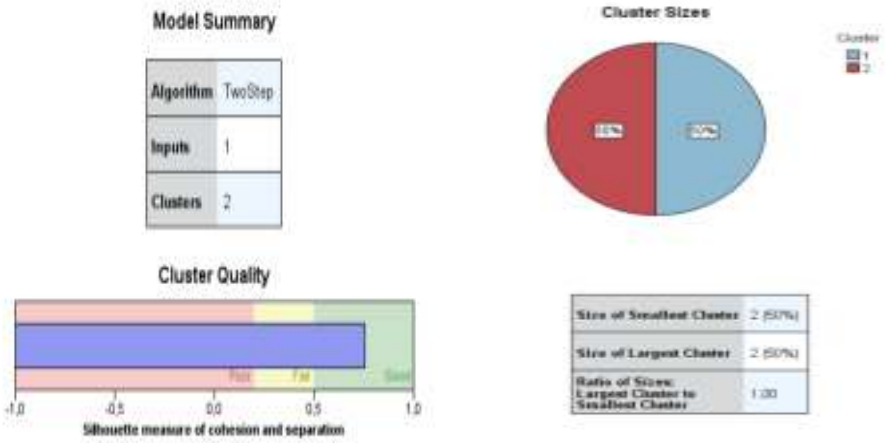


Figure 3: GDP growth rate under cluster analysis

Source: personal contribution using IBM-SPSS software

The cluster's quality is good enough (0.75) and the ratio of the cluster sizes is 1.0. These results confirm the viability of the cluster approach in the paper. The unemployment rate achieved high rates during the crisis' impact. The economic recovery started in 2010 led to a decrease of the unemployment (see Table 2).

Table 2. Unemployment rate (%)

	2009	2010	2011	2012	2013	2014	2015
EU28	8.1	9.7	9.7	10.5	10.9	10.7	10.4
Bulgaria	10.3	10.3	11.3	12.3	12.9	12.7	12.1
Romania	6.4	7.3	7.4	7.0	7.2	7.2	7.1
Moldova	6.0	8.3	5.9	5.7	5.1	4.8	4.7

On the other hand, Moldova will achieve the lowest unemployment rates of the analyzed group in 2014 and 2015.

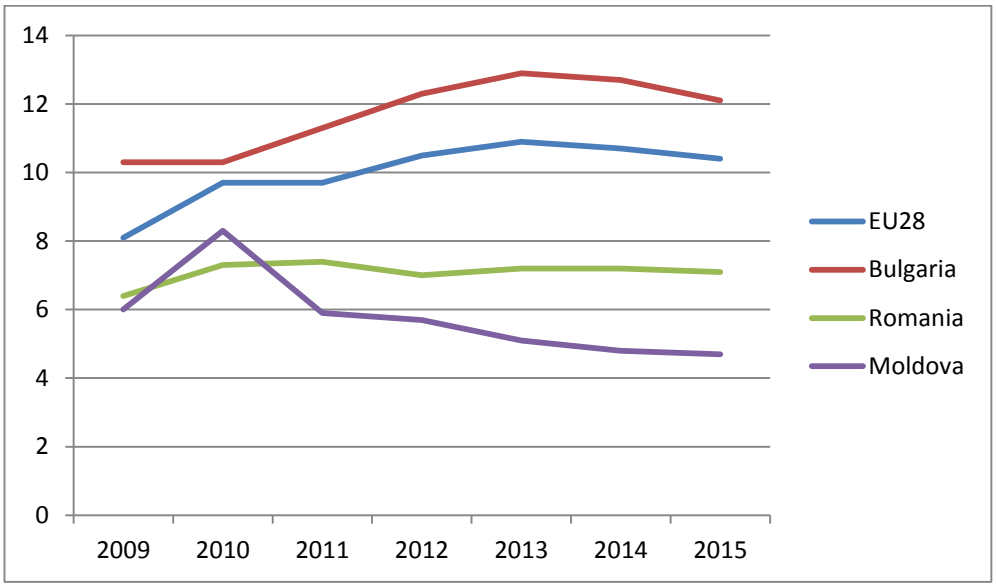


Figure 4. Unemployment rate (%)

The unemployment rate in 2014 achieved high rates in Bulgaria and EU28 and lower rates in Romania and Moldova (see Figure 5).

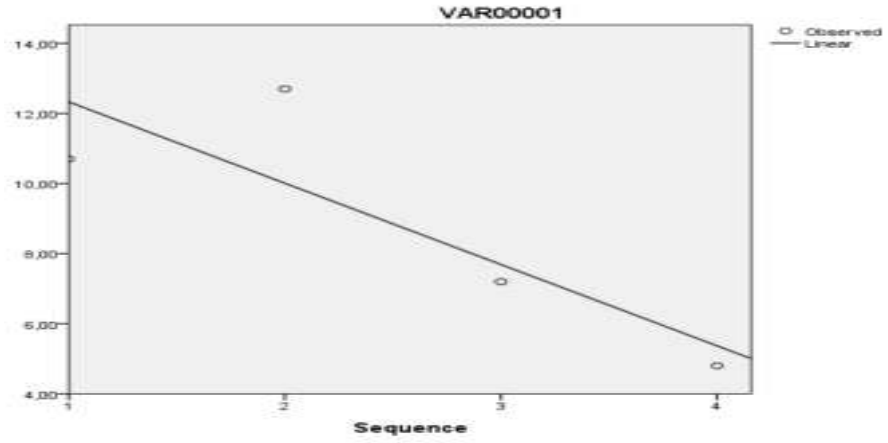


Figure 5. Unemployment rate's disparities

Source: personal contribution using IBM-SPSS software

According to the above analysis' steps, the cluster structure is presented in Figure 6.

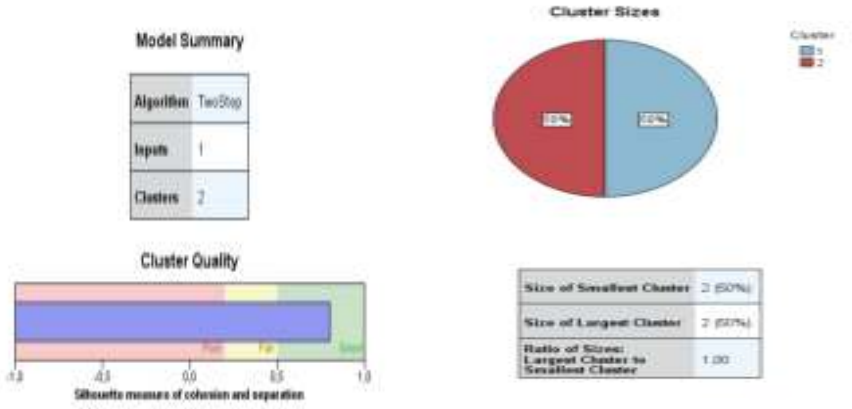


Figure 6: Unemployment rate under cluster analysis

Source: personal contribution using IBM-SPSS software

The cluster quality (0.8) is followed by a good ration of cluster sizes (1.0).

The third indicator used in the analysis is the inflation rate. All three analyzed economies faced to higher inflation rates than the EU average (see Table 3).

Table 3. Inflation rate (%)

	2009	2010	2011	2012	2013	2014	2015
EU28	1.3	1.1	2.1	1.6	0.5	0.2	0.5
Bulgaria	2.0	2.0	2.4	1.4	-0.6	-0.5	0.8
Romania	5.1	5.1	4.8	2.4	2.2	1.4	2.4
Moldova	1.2	10.3	6.2	6.6	3.5	4.5	3.9

Moldova will face to highest inflation rates during 2014-2015 (see Figure 7).

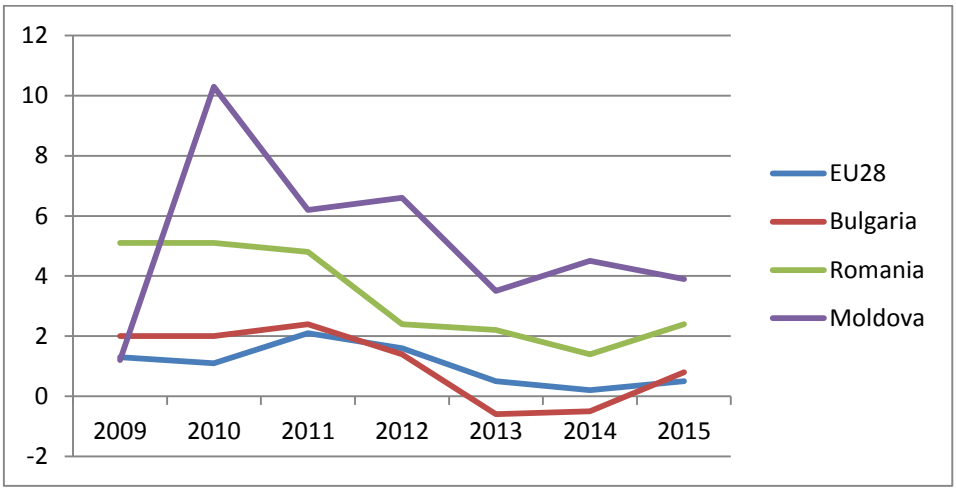


Figure 7. Inflation rate (%)

Moreover, the inflation disparities between the four economies are very high in 2014 (see Figure 8).

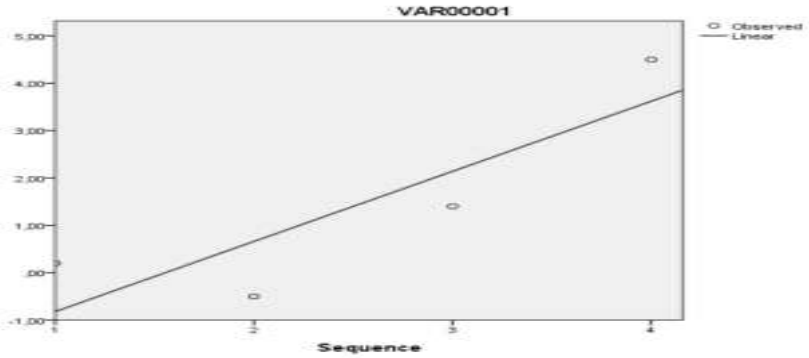


Figure 8. Inflation rate's disparities

Source: personal contribution using IBM-SPSS software

According to Figure 8, two clusters can be built under the inflation rate. The viability of this assumption is demonstrated in Figure 9. Unfortunately, a good cluster quality (0.83) is followed by a high ratio of cluster sizes (3.0).

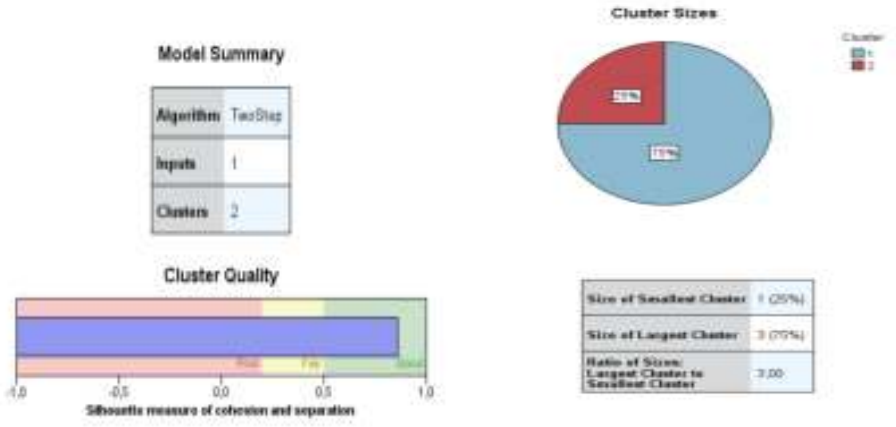


Figure 9. Inflation rate under cluster analysis

Source: personal contribution using IBM-SPSS software

The last analyzed economic indicator is public debt. This debt will be basically constant high during 2013-2015 in the EU28.

Table 4. Public debt (% of GDP)

	2009	2010	2011	2012	2013	2014	2015
EU28	80.0	80.0	82.8	86.6	89.4	89.7	89.5
Bulgaria	16.2	16.2	16.3	18.5	19.4	22.7	24.1
Romania	30.5	30.5	34.7	38.0	38.3	39.3	39.2
Moldova	29.0	31.9	30.3	33.2	32.5	33.2	33.9

Moldova has not the worst situation connected to the public debt (see Figure 10).

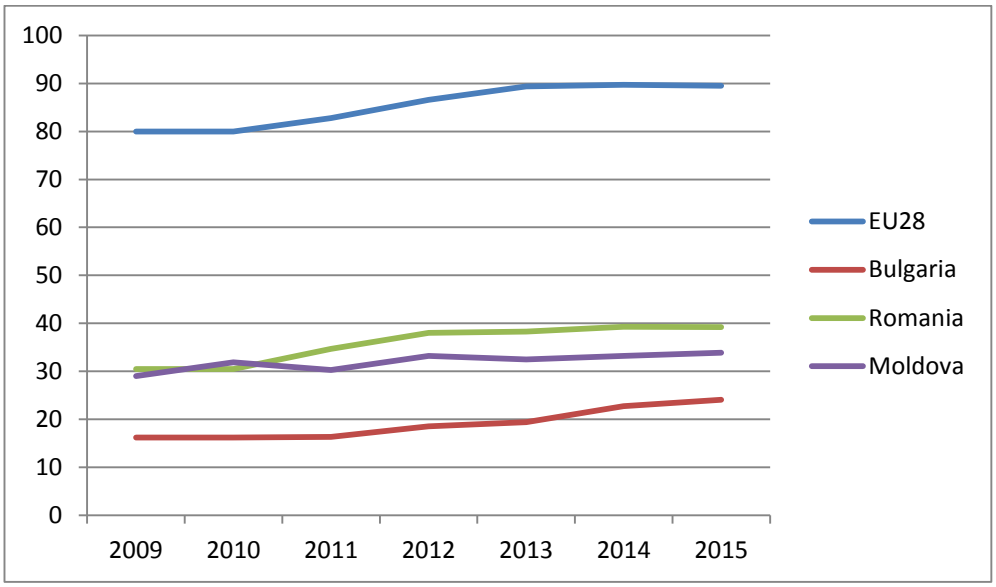


Figure 10. Public debt (% of GDP)

On the other hand, Bulgaria, Romania and Bulgaria will achieve lower public debt rates that the EU average. This situation leads to Figure 11.

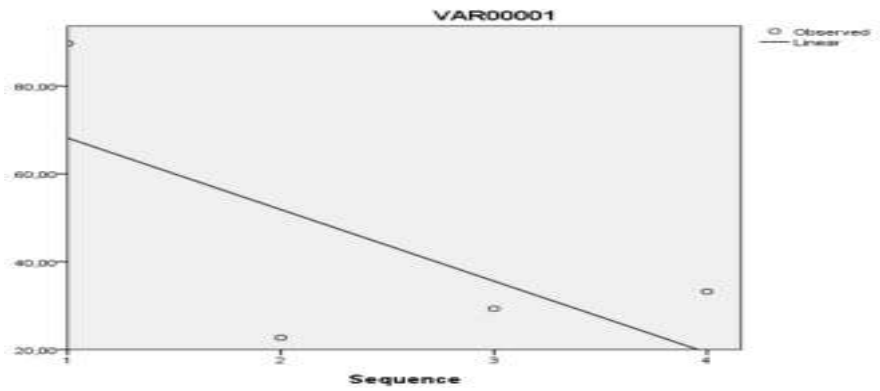


Figure 11. Public debt rate's disparities

Source: personal contribution using IBM-SPSS software

The two-step cluster approach is supported by the best cluster quality in the paper (0.9) but high ratio of cluster sizes (3.0).

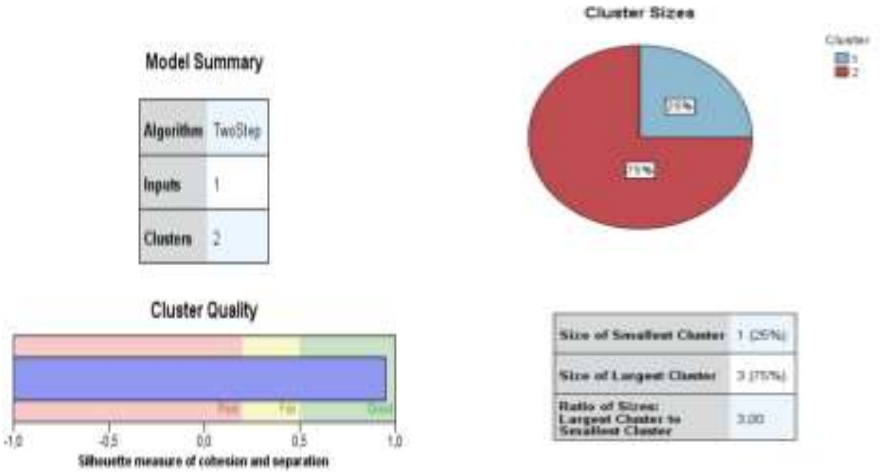


Figure 12. Public debt under cluster analysis

Source: personal contribution using IBM-SPSS software

The first intermediate conclusion in this chapter is that Moldova will achieve better performances for three indicators on short time (2014-2015). The problem is the inflation rate, which is still too high.

The second conclusion supports the cluster approach in economic analysis, which is verified by the cluster quality's values and the ration of the cluster sizes.

5. Forecasting the Moldavian Economy during the New Financial Perspective

The positive dynamics of the above three economic indicators in Moldova have to be analyzed carefully. These dynamics are not able to cover the great disparities between Moldova, Bulgaria, Romania and the EU28 average.

The GDP/capita, for example, is a good example. The latest official statistic data talk about great disparities between the four above economic entities (see Figure 13).

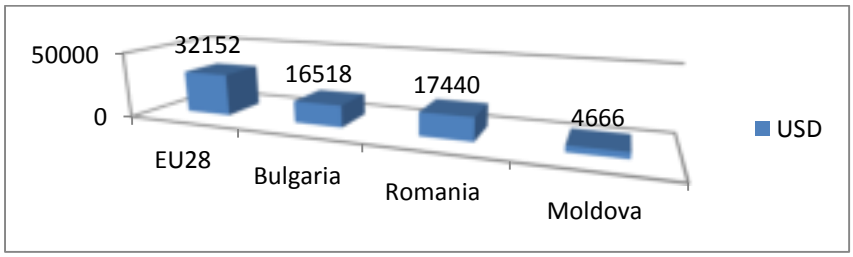
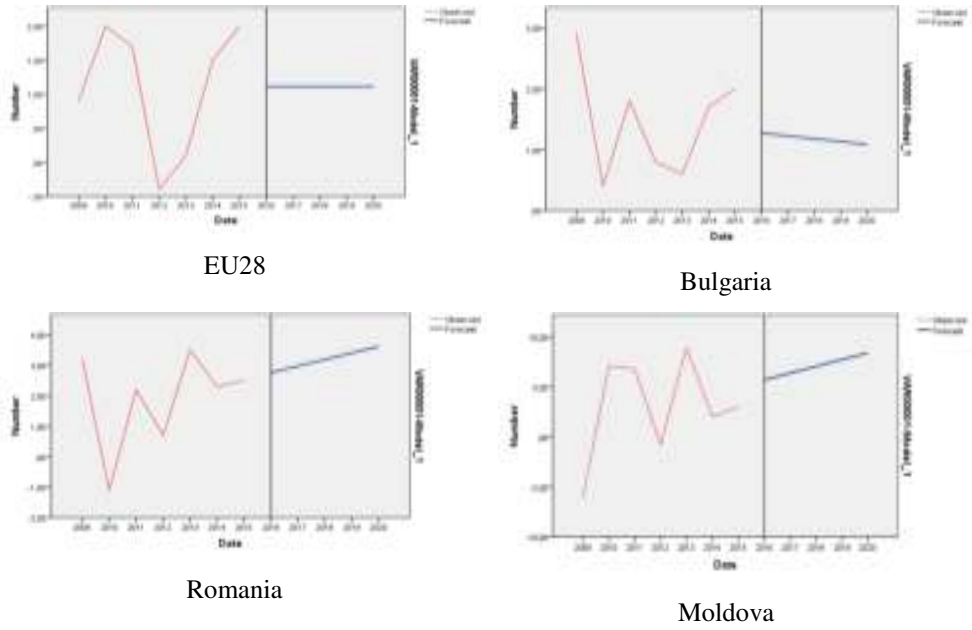


Figure 13. GDP/capita in 2013 (USD)

According to Figure 12, GDP/capita in Moldova represents only 14.51% of the average GDP/capita in EU28 (International Monetary Fund, 2014). As a result, forecasting procedures are necessary in order to see the performance of the Moldavian economy at the end of 2020. In order to obtain more available results, the period used for forecasting was extended to 2000-2014. The GDP/capita forecasting leads to the following results (see Figure 14).



EU28 Bulgaria

Romania Moldova

Figure 14: GDP forecasting (%)

Source: personal contribution using IBM-SPSS software

According to Figure 14, Moldova will achieve high GDP growth rates, especially during 2016-2020. Unfortunately, these high growth rates will not be able to eliminate the diference between the Moldavian economy and EU28 average at the end of forecasting period.

The unemployment represents a great challenge for Moldova. The unemployment rate forecasting leads to positive results (see Figure 15).

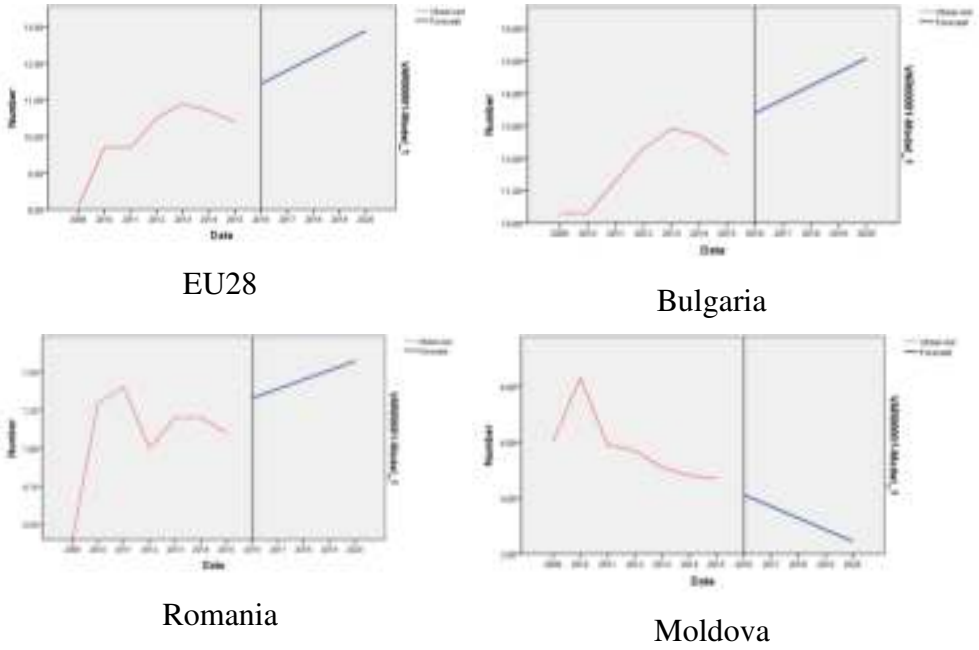
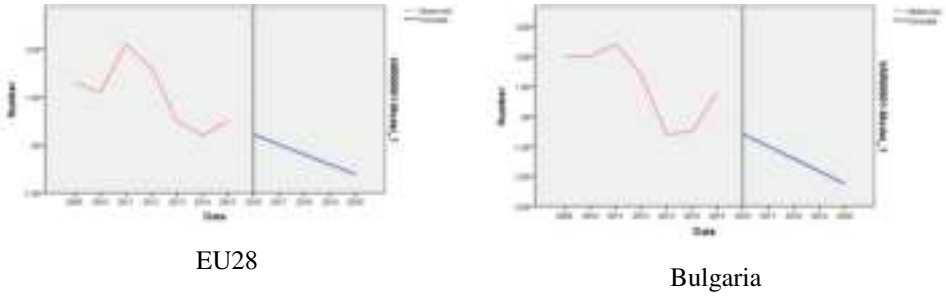


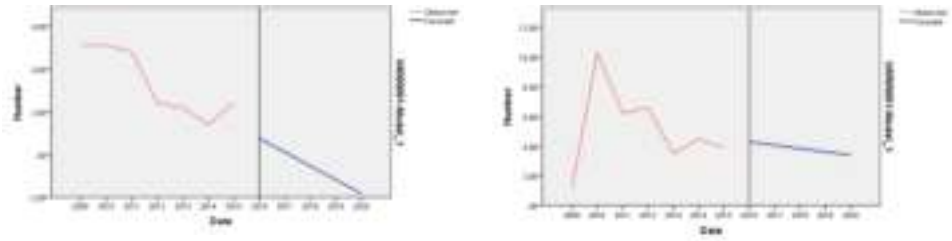
Figure 15. Unemployment rate forecasting (%)

Source: personal contribution using IBM-SPSS software

Basically, Moldova will achieve the best results related to the unemployment rate in 2020. This is a positive point, able to support its future adhering to the EU.

The inflation rate is important in having a competitive economy. Moldova faced to high inflation rates. Nowadays, it succeeded in decreasing the inflation rate but not enough. The trend of this indicator is inadequate (see Figure 16).



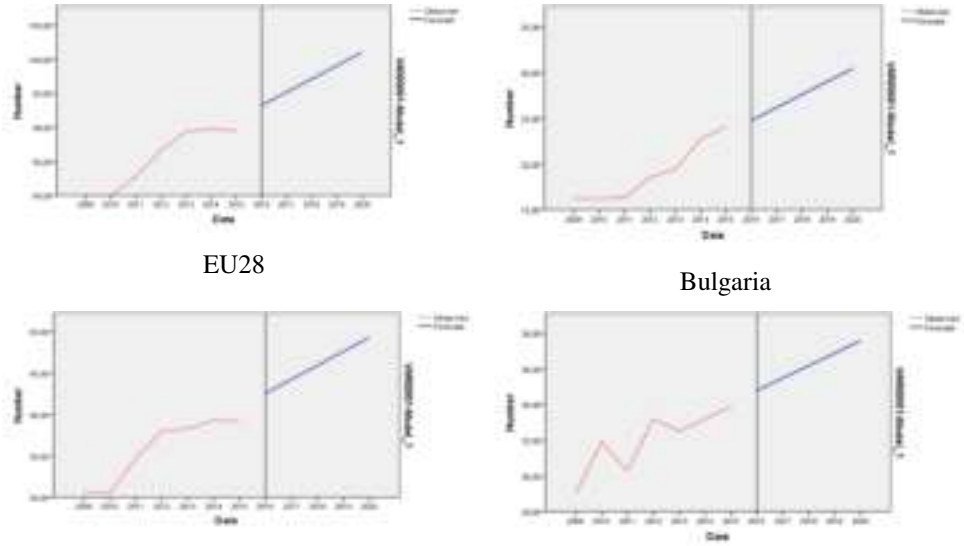


Romania Moldova

Figure 16. Inflation rate forecasting (%)

Source: personal contribution using IBM-SPSS software

Moldova will face to high inflation rates, even in 2020, while the other three economic entities will pass through disinflation. As a mechanic model, the forth economic indicator seems to have better values, at least in 2014.



EU28 Bulgaria

Romania Moldova

Figure 17. Public debt forecasting (% of GDP)

Source: personal contribution using IBM-SPSS software

With a forecasted public debt of 37% of GDP, Moldova will have a better position than the EU average in 2020.

6. Conclusions

The military and political crisis in Ukraine, the new elections' results, the permanent support of Romania and Moldova's history ask for the European way. The analysis of the Moldavian economy leads to less positive conclusions. Despite the high efforts to decrease the socio-economic disparities between Moldova and the Member States, the forecasts are not optimistic. Even in 2020, Moldova will be not able to achieve the EU average economic performances. A realistic adhering process has to take into account at least a medium term. Until then, EU, including Romania, has to improve the financial and technical support for Moldova. Finally, a political decision can be used in order to accelerate Moldova's adhering to the EU.

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A New Approach to Utility Function

Cătălin Angelo Ioan[1]

Abstract: This paper treats from an axiomatic point of view the notions of indifference and preference, relative to the consumption of goods. After the introduction and analysis of indifference classes, the notion of the utility function is introduced naturally, a number of axioms giving consistency and rigor. The concept of marginal utility is presented in both differentiable and, especially, in the discretized case. There are introduced new types of discretized marginal utility that adapts better when analyzing discrete the differential situation. The marginal rate of substitution is addressed globally, for n goods, obtaining the notions of hyperplane or minimal vector of substitution. Also, in the discretized case, there are introduced the marginal rates of substitution to the left, right or bilateral, as well as the adjusted rates, which give more precisely the possibility of consumption of a good when replaced another.

Keywords indifference; preference utility; marginal rate of substitution

Jel Classification: D01

1. Introduction

From moments of impasse that has passed mathematics at the end of the nineteenth century, when it was forced relocation and reconstruction of the foundations of rigorous, any scientific theory that any aims to be sustainable and, above all, rigorous, has a urge to be built on solid bases, axiomatized. This field theory make a distance from the field of speculations or circumstantial situations, giving durability and at the same time, rigorous scientific reasoning. Any economic activity involves the existence of two distinct entities, but complementary, namely at least one manufacturer and a single consumer.

A manufacturer can not operate without a specific guarantee the possibility of purchase of his goods by at least one buyer as such it can not exist an applicant without the real creator of the product to be asked. It is natural to assume that each of the two parties follows a well-defined purpose. A manufacturer which would not pursue its profit maximization (even if this approach is somewhat simplistic) were closer to a charitable institution, rather than an economic entity. On the other hand, a beneficiary (which departs net from the notion of consumer) that would purchase

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products without having to follow a specific purpose (food needs, comfort, travel, etc.) could easily turn into a collector therefore, it would affect another individual needs.

It is very difficult to be measured or quantified the consumer's "need". A concept, largely controversial, satisfying to some extent, provided the "primary concept" in any axiomatic theory, is the **utility**.

There are a number of theories that define, more or less axiomatic notion of utility directly related to consumer preference for certain combinations of assets. What is a consumer preference but a good over another? The answer can only return to the same concept just try to explain it. We will not deviate too much from this line (even it is questionable from many points of view), but we will try a systematic and consistent increase in scientific endeavor.

2. Consumer Preferences

Before defining the consumer space, we consider first, that all goods for consumption are indefinitely divisible. We will see, a little later, that this convention is to a point, benefit, meaning that differential techniques can be applied in the analysis of consumer behavior. On the other hand, the findings obtained will be applied with great caution, especially when we want to establish a consumer verdict.

We thus define the **consumer space** on $\mathbf{R}^n$ for n fixed assets as $SC=\{(x_1,...,x_n) \mid x_i \geq 0, i=\overline{1,n}\}$ where $x \in SC$, $x=(x_1,...,x_n)$ is a **consumption basket** or **basket of goods**.

In relation to the issues raised above, is a natural question: considering two elements $x,y \in SC$, how do we characterize that a consumer will choose the basket x or y? It seems then that will have to establish a certain choice between a basket or another. In order not to enter the above vicious circle, we define the so-called preference relations, external in the generation of the rigorous theory, but effective in implementation.

We will define the **relationship of indifference** on SC noted, in what follows, with: $\sim$. If two baskets x and y are in relation $x \sim y$, this means that any combination of goods x and y is indifferent for the consumer. Also, we note that $x \not\sim y$ the fact that x is not indifferent to y.

We will impose the condition of indifference to be a relationship of equivalence that is:

I.1. $\forall x \in SC \Rightarrow x \sim x$ (reflexivity);

I.2. $\forall x,y \in SC$, $x \sim y \Rightarrow y \sim x$ (symmetry);

I.3. $\forall x,y,z \in SC$, $x \sim y$, $y \sim z \Rightarrow x \sim z$ (transitivity).

The interpretation of these axioms is natural. Thus, reflexivity is not merely say that a basket of goods is indifferent in his choice himself, and symmetry that indifference between x and y implies choice, inevitably, the indifference of y and x. Transitivity is not always obvious, in that there may be situations (more or less forced) the indifference between x and y, then y and z between not involve the binding of x and z. Usually, the deviation from transitivity can occur when the relation of indifference is not "perfect", small differences between the three baskets leading to a significant distance between the extremes.

Let therefore the consumer space endowed with the relationship of indifference defined upper (SC,~) and $x \in SC$. The equivalence class of x: $[x]=\{y \in SC \mid y \sim x\}$ will consist in all consumer's baskets indifferent respected to x. We will call [x] – **the indifference class** of x.

From the properties of equivalence classes follow some remarkable conclusions, namely:

- x and y are indifferent if and only if they have identical indifference classes;
- for any two baskets of goods x and y, their indifference classes are either identical or disjoint (i.e. if exists z such that x~z, but $y \not\sim z$ then for any u~x will result that $u \not\sim y$;
- the set of all baskets of goods or, in other words, the **consumer space** is the union of all classes of consumer indifference.

A system of representatives for the relationship of indifference ~ will consist of all consumer baskets such that any two such entities are not indifferent and any consumer basket are whatever exactly one of the elected representatives.

Before continuing, let recall that a norm on the linear space $\mathbf{R}^n$ is an application: $\|\cdot\| : \mathbf{R}^n \to \mathbf{R}$, $x \to \|x\|$ $\forall x \in \mathbf{R}^n$ such that the following axioms are satisfied:

N.1. $\|x\|=0 \Rightarrow x=0$;

N.2. $\|\alpha x\| = |\alpha| \cdot \|x\|$ $\forall x \in \mathbf{R}^n$ $\forall \alpha \in \mathbf{R}$;

N.3. $\|x + y\| \le \|x\| + \|y\|$ $\forall x,y \in \mathbf{R}^n$.

The pair $(\mathbf{R}^n, \|\cdot\|)$ is called the normed n-dimensional linear space.

Considering therefore an arbitrary norm $\|\cdot\|$ on $\mathbf{R}^n$ we will add a first additional axiom to the relationship of indifference named **the axiom of continuity**:

I.4. $\forall x,y \in SC$, $x \sim y$, $\|x\| < \|y\| \Rightarrow \exists z \in SC$ such that: $x \sim z$ and $\|x\| < \|z\| < \|y\|$.

The axiom of continuity, not simply say that switching to a basket of goods to another indifferent with it, is done continuously, without "jumps".

For $x \in SC$, we will call, in the assumption of continuity axiom, the indifference class of x as **the indifference hypersurface** or for n=2 – **the indifference curve**. Because the indifference classes are either identical or disjoint, we have that the intersection of indifference hypersurfaces or curves are impossible.

A second additional axiom of indifference with respect to the above relationship refers to the condition of the lower bound of indifference classes namely:

I.5. $\forall x \in SC \Rightarrow \exists u \sim x$ such that $\|u\| \leq \|v\| \ \forall v \sim x$

The axiom I.5 describes the condition that in a class of indifference to be a basket at a least "distance" of origin or, in other words, with the lowest total (with respect to the norm) number of goods in his structure.

We will call a basket of goods like in the axiom I.5 - **minimal basket in the sense of norm with respect to the indifference class** of $x \in SC$ and we will note m(x). It should be noted that we do not necessarily guarantee the uniqueness of the existence of such a basket, but his norm is really unique.

Moreover, if $x \sim y$ then $\|m(x)\| = \|m(y)\|$. Indeed, if $x \sim y$ then $\|m(x)\| \leq \|v\| \ \forall v \sim x$ so , in particular: $\|m(x)\| \leq \|m(y)\|$ because $m(y) \in [x]$ and hence $m(y) \sim x$. Analogously, $\|m(y)\| \leq \|m(x)\|$ hence the above statement.

To define the relationship of preference, we will formulate differently the problem. If a basket of goods will be some x preferred to another y, it is logical to assume that any other basket z indifferent to x will also preferred to y. Therefore, we will consider instead of SC, the factor set SC relative to ~ which consists in the indifference classes of SC, denoted with SC/~.

Thus, to define **the relationship of the classes** marked in the following with $\succeq$ through the following axioms:

P.1. $\forall [x] \in SC/\sim \Rightarrow [x] \succeq [x]$ (reflexivity);

P.2. $\forall [x],[y] \in SC/\sim$, $[x] \succeq [y]$, $[y] \succeq [x] \Rightarrow [x]=[y]$ (antisymmetry);

P.3. $\forall [x],[y],[z] \in SC/\sim$, $[x] \succeq [y]$, $[y] \succeq [z] \Rightarrow [x] \succeq [z]$ (transitivity)

We will impose to this relationship four additional axioms:

P.4. $\forall x,y \in SC \Rightarrow [x] \succeq [y]$ or $[y] \succeq [x]$ (the condition of total ordering);

P.5. $\forall x \in SC \Rightarrow \exists y \in SC$ such that $y \neq x$ and $[y] \succeq [x]$;

P.6. $[x] \succeq [y]$ if and only if $\|m(x)\| \geq \|m(y)\|$ (the condition of the compatibility with the existence of minimal baskets);

P.7. $\forall x,y \in SC$, $x>y \Rightarrow [x] \succeq [y]$ and $x \neq y$ (the condition of the compatibility with the strict inequality relationship).

At first glance, the relationship of preference seems to depart from the nature of the goods, operating on indifference classes which represents set of goods indifferent between them.

On the other hand, however, the advantage of considering the indifference classes lies from the axiom P.2 which with P.1 and P.3 give the order relation character. Otherwise, if the relationship would be defined strictly preferably baskets of goods, from the fact that the other one was preferred and the second to the first, result that they are not identical, but that they are indifferent, so just classes equal indifference.

The total ordering condition states that any two baskets of goods are comparable in meaning preference for one of them.

The P.5 axiom guarantees the existence for any basket of goods of one not indifferent with it with and be at least as much preferred the former.

The axiom P.6 states that a class is preferred to another if and only if the norm of the first basket is greater than or equal to those of the minimal basket of the second.

The P.7 axiom states that a small additional quantity of a good from a basket leads to a preference superior to the original. The axiom also shows the existence for any basket of goods of one superior relative to the preference and, analogously, lower like preference.

Let now $x \in SC$ and $m(x)=(\bar{x}_1,...,\bar{x}_n) \in [x]$.

We will define **the relationship of preference** noted, in the following, without the danger of ambiguity, also with $\succeq$, by: $\forall x,y \in SC$ $x \succeq y$ if and only if $[x] \succeq [y]$.

The relationship will keep the properties of reflexivity, transitivity and total ordering, but the antisymmetry becomes:

P.2'. $\forall x,y \in SC / \sim$ $x \succeq y$, $y \succeq x \Rightarrow x \sim y$

The interpretation of the axioms is obvious. If any indifference class of a basket of consumer is preferred at least as much itself (reflexivity), follows that for an arbitrary basket x, any basket y indifferent to x, will be at least as preferred much as x.

The symmetry states that if a basket z indifferent to x and a basket t indifferent to y are each preferred by at least as much the other, then the two consumer baskets belonging to the same classes, so they are indifferent between them.

The transitivity can have violations in practice, but will usually be excluded from the analysis. It is possible that if relations preferably slightly offset time or applied to different situations, not to achieve transitivity. To have ensured transitivity, it must first be satisfied the simultaneity of the moments of choice and, on the other hand, it must apply to the same circumstantial situation.

We will define now on SC **the strict preference relationship** as a relationship class, denoted $\succ$ and defined by $[x]\succ[y]$ if and only if $[x]\succeq[y]$ and $[x]\neq[y]$ (which is equivalent to $[x]\cap[y]=\varnothing$).

Similarly, we now define **the strict preference relationship** denoted in the following, without the danger of confusion, by $\succ$: $\forall\, x,y\in SC\ x\succ y$ if and only if $[x]\succ[y]$.

The strict preference relationship on classes is not, obviously, reflexive because if for $[x]\in SC/\sim$ implies $[x]\succ[x]$ then $[x]\neq[x]$ which is a contradiction. Also, the antisymmetry states that: $\forall[x],[y]\in SC/\sim\ [x]\succ[y],\ [y]\succ[x]\Rightarrow[x]=[y]$. But $[x]\succ[y]$ implies $[x]\neq[y]$ so a contradiction with the statement of conclusion. Relative to transitivity, if $\forall[x],\ [y],\ [z]\in SC/\sim,\ [x]\succ[y],\ [y]\succ[z]$ implies $[x]\succeq[y],\ [x]\neq[y\ ],\ [y]\succeq[z],\ [y]\neq[z]$ therefore: $[x]\succeq[z]$. The fact that $[x]\neq[z]$ does not result from any assertion, therefore it can not be proven the transitivity in this axiomatic framework.

For this reason, we will use below only indifference or preference relations (not strictly), in order to make full use of "power" property of the relations of equivalence or order.

Consider now an arbitrary consumer basket $x\in SC$. We will call **the preferred area of consumer** of x, the set: $ZC(x)=\{y\in SC \mid [y]\succeq[x]\}$ that is the set of those baskets that consumer prefers at least as much of x.

It notes that under the axiom P.5, $ZC(x)-[x]\neq\varnothing$ that is in the preferred are of consumption of x is at least one basket y strictly preferred to x.

Let us note that if $y\in ZC(x)$ then for any $z\in ZC(y)$ we have: $[z]\succeq[y]\succeq[x]$ from where, by virtue of transitivity: $[z]\succeq[x]$ so $z\in ZC(x)$. Therefore:

$$\forall y\in ZC(x)\Rightarrow ZC(y)\subset ZC(x)$$

From the axiom P.5 $\exists z\in SC$ such that $z\neq x$ and $[z]\succeq[x]$. From the above results we have that $ZC(z)\subset ZC(x)$. It is clear that $x\notin ZC(z)$, otherwise having $[x]\succeq[z]$ and from antisymmetry, results $[x]=[z]$ so $x\sim z$ – contradiction. After this observation we have that for any $x\in SC\ \exists y_1\in ZC(x)$ such that $ZC(y_1)\subset ZC(x)$, $ZC(x)-ZC(y_1)\neq\varnothing$ (that is the

inclusion is strictly). Analogously $\exists y_2 \in ZC(y_1)$ such that $ZC(y_2) \subset ZC(y_1)$, $ZC(y_1)-ZC(y_2) \neq \emptyset$.

Therefore, for any $x \in SC$ $\exists (y_n)_{n \geq 1} \subset SC$ such that:

$$ZC(x) \supset ZC(y_1) \supset ZC(y_2) \supset ... \supset ZC(y_n) \supset ...$$

the inclusions being strictly, so the underlying consumption of some basket, contains an infinity of different baskets.

Examples

1. Considering any two goods, the relationship $x \sim y$ defined by: $ax_1+bx_2=ay_1+by_2$ $\forall x=(x_1,x_2)$, $y=(y_1,y_2) \in SC$ where $a,b>0$ is one of indifference. The indifference classes relative to $\sim$ are for any $x=(x_1,x_2)$: $[x]=\{y \in SC \mid ay_1+by_2=ax_1+bx_2\}$

2. Considering any two goods and the indifference relation defined in the first example, the relationship $[x] \succeq [y]$ defined by: $ax_1+bx_2 \geq ay_1+by_2$ $\forall x=(x_1,x_2) \in [x]$, $y=(y_1,y_2) \in [y] \in SC$ where $a,b>0$ is a preference relationship. Considering now $x \in SC$, $x=(x_1,x_2)$, $ax_1+bx_2=U$, we have $ZC(x)=\{(y_1,y_2) \mid ay_1+by_2 \geq U\}$.

At the end of this section, we ask the normal question: how can we define concretely in practice, the relations of indifference or preference?

A first approach would be the income of the consumer is willing to spend on a basket of some goods. Considering two baskets of goods $x=(x_1,...,x_n)$ and $y=(y_1,...,y_n)$ we can believe that $x \sim y$ if a consumer is willing to devote the same amount of money for the purchase of x and y, respectively. The problem of preference is much more complicated. Considering the amount of the money S that the consumer is willing to spend to purchase a basket of goods (with some fixed structure) we could say that $x \succeq y$ if the sum S_x necessary to obtain x is greater than or equal to the corresponding S_y for the purchasing y, both amounts being less than or equal to S. This type of choice is quite limited but its concrete applicability. On the one hand, even if the price of a particular good would be identical to the market (otherwise, the consumer could purchase basket of goods from various sources and the relationship of preference could be, in some cases, reverse to the income allocated) the internal structure basket could lead to situations of exclusion in certain parts of it.

Consider, for example, a customer that has disposable an income of 12 monetary units wishing to purchase two products, namely bread whose price is 3 u.m./pcs. and toothpaste with the price 5 u.m./pcs. Considering pairs of goods of the form (p,d) where p - number of breads and d - the number of tubes of toothpaste, all baskets will be made admissible in pairs: (0,1) - 5 u.m., (0,2) - 10 u.m., (1,0) - 3 u.m., (1,1) - 8 u.m., (2,0) - 6 u.m., (2,1) - 11 u.m., (3,0) - 9 u.m., (4,0) - 12 u.m. The consumption basket that will surpass all others will contain, from this point of view, 4 breads and

no toothpaste. The consumer then allocated the entire amount available, but satisfaction does not seem in any case, being the greatest because, on the one hand, did not buy any toothpaste (which had actually needed), and on the other, bought four breads that, if he lives alone, could be much more than its food needs. In the idea that he can not eat more one bread per day, most rational choice would be (1,1), but not willing to be spent maximizing income! Another choice that would ensure the two products could be (2,1) but, again, would bring an extra supply of breads which may not need them.

We see therefore that, in principle, the space of consumption SC should be limited according to consumer needs. On the other hand, a strict monetary approach to consumer preferences may lead to extreme situations that cause, in fact, dissatisfaction.

3. The Convexity of the Areas of Consumption

Considering a set A in $\mathbf{R}^n$ this is called convex if $\forall x,y \in A\ \forall \lambda \in [0,1] \Rightarrow \lambda x+(1-\lambda)y \in A$. Considering the line segment passing through points M(x) and N(y) we have that a set is convex if the segment MN (noted also [x,y]) is entirely in it.

In particular, we assume in what follows, that for any consumption basket $x \in SC$, ZC (x) is a convex set.

What significance has this fact and where it is the origin for this restriction?

The problem is quite complicated and, at first glance, seems somewhat common sense to take this restriction. If $y,z \in SC(x)$ then y and z are preferred to x. It seems natural to believe that any combination of intermediate goods between y and z will be preferred to x.

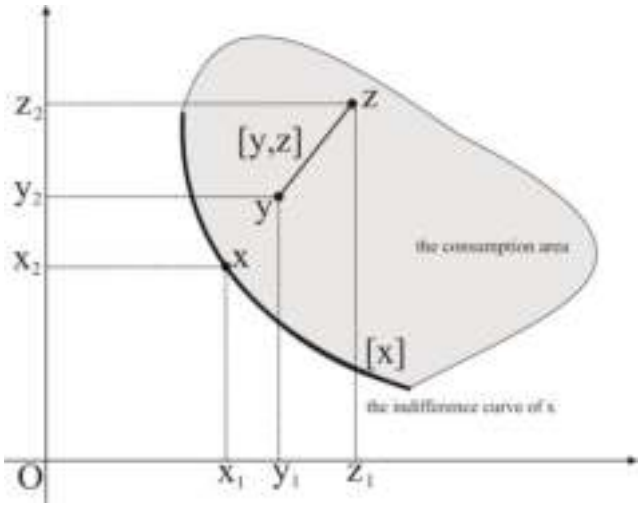

Figure 1. The convexity of the consumption area

Unfortunately, not always so! Consider, as an example a person who wants to travel to work effectively. If x="walking", then y="bus travel" and z="travel by cab" will be the preferred choices of x (ignoring here the actual distances or transport costs). A combination of y and z will be, for a relatively short distance, always disadvantageous to the first, because waiting times flowed into the travel mode change.

We believe however that most of the areas of consumption is convex, for several reasons. On the one hand, a basket of goods x, generating non-convex consumer area, is too unstable to be taken into account by a "rational" consumer. Any combination of consumer goods within the area can lead potentially to a reduction of its satisfaction with respect to x. On the other hand, even if our analysis is static, in reality, the migration is dynamic (it takes place in some time) and it is hard to believe that the consumer will go through a period of consumer dissatisfaction reach, for example, from y to z.

Formalizing, we will say that $\forall x \in SC$, ZC(x) is a convex set that is $\forall y,z \in SC$ such that $y \succeq x$ and $z \succeq x$ follows $\lambda y+(1-\lambda)z \succeq x$ $\forall \lambda \in [0,1]$.

From the condition of convexity of ZC, we get that for any $x,y,z \in SC$ such that $[y] \succeq [x]$ and $[z] \succeq [x]$ then $[\lambda y+(1-\lambda)z] \succeq [x]$ $\forall \lambda \in [0,1]$.

4. The Utility Functions

In the previous section, we have noted the difficulty of the mathematical approach of indifference and preference concepts. We will try in this part an axiomatic introducing of a concept, even though disputed by many economists, will bring some light in the treatment of previous notions. For a mathematical analysis of efficient consumer preferences could be useful to introduce a function with numerical values to enable their hierarchy.

We thus define **the utility function** as:

$$U:SC\rightarrow\mathbf{R}_+,\ (x_1,...,x_n)\rightarrow U(x_1,...,x_n)\in\mathbf{R}_+\ \forall(x_1,...,x_n)\in SC$$

satisfying the following axioms:

U.1. $\forall x,y\in SC:\ x\sim y \Leftrightarrow U(x)=U(y)$;
U.2. $\forall x,y\in SC:\ x \succeq y \Leftrightarrow U(x)\geq U(y)$;
U.3. $U(0)=0$

We can reformulate the definition of utility function in terms of indifference classes as follows:

$$U:SC/\sim\cup\{0\}\rightarrow\mathbf{R}_+,\ [x]\rightarrow U([x])\in\mathbf{R}_+\ \forall[x]\in SC/\sim$$

satisfying the following axioms:

U.1'. $\forall x,y\in SC/\sim:\ [x]=[y] \Leftrightarrow U([x])=U([y])$;
U.2'. $\forall x,y\in SC/\sim:\ [x] \succeq [y] \Leftrightarrow U([x])\geq U([y])$;
U.3'. $U(0)=0$

We note that axiom U.1' does not mean anything other than injectivity of the utility function on the set of indifference classes.

Analyzing carefully the definition of utility, we see that, in fact, it brings nothing new concept in relation to the relations of indifference or preference.

Indeed, considering an arbitrary function, strictly increasing (with respect to the non-total order relation $\geq$), $U:SC\rightarrow\mathbf{R}_+$, we can define on SC the relationship of indifference as: $\forall x,y\in SC:\ x\sim y\Leftrightarrow U(x)=U(y)$. The relationship satisfies the axioms I.1, I.2 and I.3 of the previous definition of indifference. We can also define the relationship of preference by: $\forall x,y\in SC/\sim:\ [x] \succeq [y]\Leftrightarrow U([x])\geq U([y])$. The axioms P.1, P.2, P.3, P.4 and P.5 are also satisfied.

From the axiom P.7 we have that if $x,y\in SC$ such that $x>y$ then $[x] \succeq [y]$ therefore $U(x)>U(y)$. The utility function is therefore strictly increasing relatively to the relationship of strictly inequality. Let us note however that due to the impossibility

of defining a relationship of total order on $\mathbf{R}^n$ we can not speak of a strict monotony of the overall definition scope.

Under the two definitions, we can characterize the class of indifference relative to a basket $x \in SC$ like $[x]=\{y \in SC \mid U(y)=U(x)\}$ and the consumer's area of x: $ZC(x)=\{y \in SC \mid U(y) \geq U(x)\}$.

Consider now $x \in SC$ and $U(x)=a \in \mathbf{R}_+$. We have therefore:

$$[x]=\{y \in SC \mid U(y)=a\}$$

If $y,z \in [x]$ then: $U(y)=U(z)=a$. We have seen, above, from the convexity of ZC(x) that: $[\lambda y+(1-\lambda)z] \succeq [x]$ or, in terms of utility: $U(\lambda y+(1-\lambda)z) \geq U(x)=a=\lambda a+(1-\lambda)a=\lambda U(y)+(1-\lambda)U(z)$.

We obtained thus:

$$U(\lambda y+(1-\lambda)z) \geq \lambda U(y)+(1-\lambda)U(z) \ \forall \lambda \in [0,1] \ \forall z,y \in [x] \ \forall x \in SC$$

The above condition is nothing but than the concavity of a function. In the case of a continuous function, the concavity is expressed geometrically by the fact that any chord determined by two points on the graph function is located below it. Therefore, the restriction of the utility function to a class of indifference of an arbitrary basket is concave.

We will extend this requirement to the whole space SC, thus requiring the utility function the following condition:

UC.1. The utility function is concave.

While not necessarily essential to the fundamental properties, we must sometimes still an additional condition:

UC.2. The utility function is of class C^2 on the inside of SC.

The differentiability of the utility function automatically implies its continuity on the interior domain of definition. In the case UC.2 the concavity axiom that function is equivalent to the fact that the second differential of U is defined negatively.

Considering $d^2U=\sum_{i,j=1}^{n} \frac{\partial^2 U}{\partial x_i \partial x_j} dx_i dx_j$ and the attached quadratic form: $H=\sum_{i,j=1}^{n} \frac{\partial^2 U}{\partial x_i \partial x_j} h_i h_j$, the fact that H is negatively defined is shown by Gauss method or by that of Jacobi.

Also, in the case of the differentiability, let note that $\sum_{i=1}^{n}\left(\frac{\partial U}{\partial x_i}\right)^2 \neq 0$ that is at least one of the first order partial derivatives is nonzero at any point. Indeed, if there is a point such that: $\frac{\partial U}{\partial x_i}=0\ \forall i=\overline{1,n}$ then, from the concavity of U it follows that it is a local maximum. On the other hand, the axiom P.7 imposed the hypothesis of non-existence of local maximum or minimum points.

We can not conclude this section without a perfectly legitimate question: how we will effectively build the utility function?

In principle, we can assign arbitrary values to the indifference classes, which will be satisfied only condition being that if x is preferred to y then the value assigned to the class of x to be equal to or greater than that attributed to class y. In this case, the detailed rules for the award is very relative.

If we are not interested than order of preference for a basket of goods and another, serial numbers can be assigned arbitrarily (e.g. order of preference indexed by non-zero natural numbers), that will do a hierarchy of the baskets of goods. In this case, we say that we are dealing with an **ordinal utility**.

Its disadvantage is, on the one hand, that we have not an uniqueness in assignment and, on the other hand, the utility thus defined can not be used in complex mathematical calculations (because of dependence by the arbitrary allocation).

Another way the award is related to external factors which contribute to the expression of preference for a basket of goods or another. We can define the utility for the purposes of income the consumer is willing to allocate for purchase a basket of goods. Thus, a consumer who has 7 u.m. put in a position to choose between buying a basket of soft drink with a price 2 u.m. and a sandwich of 3 u.m. and one of two drinks a 3 u.m. (together) and a sandwich for the same price he chooses, most often, the latter combination.

Also, we can define the utility as the consumer's economy that makes reference to a standard basket of goods in the choice amounts to the same invoice. We could give an example where a person is indifferent where to go in it's free time: to the theater, the cinema or a concert. If a theater ticket will cost 20 u.m., at the cinema - 10 u.m. and 30 u.m. at the concert, he will take the concert like standard and if he go to the theater will have a utility of 10 u.m. (30-20) and analogously, to the cinema - 20 u.m. (30-10).

Another approach may be of utility in terms of satisfaction in the future purchase act. Thus, an individual who is in a position to choose between a TV and a computer

having identical prices, choose the TV if it has no notions about computers and choose the computer that is definitely going to write a book about the theory of utility!

However we put the problem, it is agreed that an allocation of utility which abides the axioms and will meet the above conditions can be addressed mathematically more correct once it has been precisely defined. We call such an allocation: **cardinal utility**.

Let now a concrete way to approach the construction of utility functions.

Considering $x \in SC$, we will define $U(x)=\|m(x)\|$. The definition is correct under axiom I.5 that for any class of indifference to a basket of certain guarantees the existence of a basket of minimum norm.

From the axiom I.5, we saw that if $x \sim y$ then $\|m(x)\|=\|m(y)\|$ so $U(x)=U(y)$. Therefore, U.1 axiom is satisfied.

If $x \succeq y$ then, from the axiom P.6, we have that $m(x) \geq m(y)$ therefore: $U(x) \geq U(y)$ so just axiom U.2.

Considering now a utility function $U:SC \to \mathbf{R}_+$ and an application monotonically increasing $f:\mathbf{R}_+ \to \mathbf{R}_+$, the function $f \circ U$ defined by $f \circ U(x)=f(U(x))$ is also an utility function. Indeed, if $x \sim y$ then $U(x)=U(y)$ from where $f(U(x))=f(U(y))$ and if $x \succeq y$ then $U(x) \geq U(y)$ and $f(U(x)) \geq f(U(y))$. We therefore conclude that the utility function is determined up to a monotone increasing application.

Finally, let mention that for $a \in \mathbf{R}_+$, the graph corresponding to the equation solutions $U(x)=a$ is called isoutility curve (in $\mathbf{R}^2$) or isoutility hypersurface (in $\mathbf{R}^n$).

From [3] and the fact that U is a concave function and partial derivatives of first order are positive (as we shall see later), it follows that the isoutility hypersurfaces are convex.

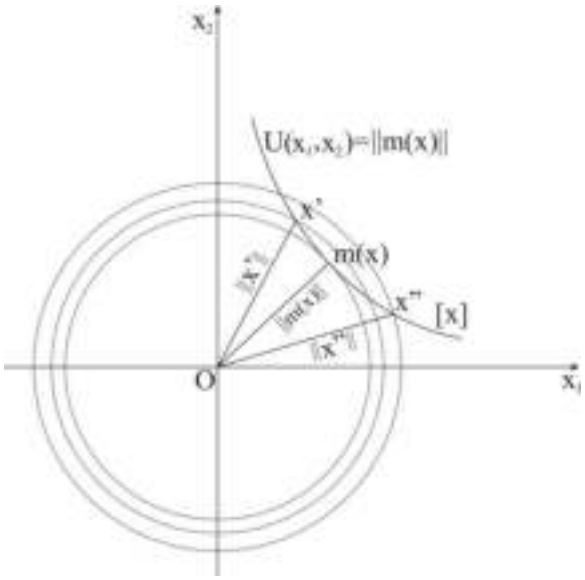

Figure 2. The definition of the utility function

Let consider now n classes of basket of goods whose consumption spaces are $SC_1 \subset \mathbf{R}_+^{k_1}$,...,$SC_n \subset \mathbf{R}_+^{k_n}$ and $U_1,...,U_n$ – corresponding utility functions. We will call the n classes **independent in the sense of utility** if the function $U:SC_1 \times ... \times SC_n \to \mathbf{R}_+$, $U(X_1,...,X_n)=U_1(X_1)+...+U_n(X_n)$ $\forall X_i=\left(x_{i1},...,x_{ik_i}\right) \in SC_i$, $i=\overline{1,n}$ is a utility for all goods.

In particular, n goods will call **independent in the sense of utility** if $U(x_1,...,x_n)=U_1(x_1)+...+U_n(x_n)$ $\forall (x_1,...,x_n) \in SC$.

One can easily see that if the functions $U_1,...,U_n$ are concave and of class C^2 then: $d^2U=\sum_{i=1}^{n} U_i^{''}(x_i)dx_i^2 \le 0$ therefore U is concave.

Example

Considering for any $n \ge 2$ goods the relationship of indifference $x \sim y$ defined by: $x_1^{k_1} x_2^{k_2} ... x_n^{k_n} = y_1^{k_1} y_2^{k_2} ... y_n^{k_n}$ $\forall x=(x_1,x_2,...,x_n)$, $y=(y_1,y_2,...,y_n) \in SC$, $k_1,...,k_n>0$, we will define after foregoing the utility function:

$$U(x)=\|m(x)\|=\sqrt{\sum_{i=1}^{n} k_i}\, k_1^{-\frac{k_1}{2\sum_{i=1}^{n} k_i}} ... k_n^{-\frac{k_n}{2\sum_{i=1}^{n} k_i}} x_1^{\frac{k_1}{\sum_{i=1}^{n} k_i}} x_2^{\frac{k_2}{\sum_{i=1}^{n} k_i}} ... x_n^{\frac{k_n}{\sum_{i=1}^{n} k_i}} \quad \forall x=(x_1,x_2,...,x_n) \in SC$$

5. The Marginal Utility

Let U:SC→$\mathbf{R}_+$ an utility function. We saw above that the utility is an increasing function with respect to the preference relation of goods basket and strictly increasing with respect to the relationship of strictly inequality on $\mathbf{R}^n$.

Considering $1\leq i\leq n$ – fixed and $a_k\in\mathbf{R}_+$, $k=\overline{1,n}$, $k\neq i$, we will note synthetic $x=(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)\in$ SC.

We define **the discretized marginal utility** in relation to the i-th good, while the consumption of other goods is constant as:

$$U_{m,i}(x)=\frac{\Delta U}{\Delta x_i}=\frac{U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x_i-\Delta x_i,a_{i+1},...,a_n)}{\Delta x_i}$$

therefore the variation of the utility U at the variation of the consumption of good i.

In relation to the above definition, we deduce easily:

$$\Delta U=U_{m,i}(x)\Delta x_i$$

It is necessary here to make an interesting observation! The classic definition of marginal utility essentially uses the variation of the utility function from one direction. Considering thus $\Delta x_i=h$, we get from above:

$$U_{m,i}(x)=\frac{U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x_i-h,a_{i+1},...,a_n)}{h}$$

therefore the variation at left in the point x.

If $h>0$ then the marginal utility at the point x is the change in utility of the "past" in "now" and can not be used to estimate the utility in the "future". Analogously, if $h=-s<0$ then the marginal utility at the point x becomes:

$$U_{m,i}(x)=\frac{U(a_1,...,a_{i-1},x_i+s,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)}{s}$$

and represents the variation of the utility from "present" in the "future" and can not be used to calculate the utility in the "past".

A more accurate way of calculating the marginal utility is the arithmetic mean of the marginal utility to the left and right:

$$U_{m,i}(x)=\frac{U(a_1,...,a_{i-1},x_i+h,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x_i-h,a_{i+1},...,a_n)}{2h}$$

for all points inside the space consumption, and to the left and right of it, calculating the marginal utility to right, respectively left.

In what follows, we will note the discretized marginal utility at left with $U_{ml,i}$, the discretized marginal utility at right with $U_{mr,i}$ and the discretized marginal utility two-sided with $U_{mb,i}$.

We obtain that:

- $\Delta_l U=U_{ml,i}(x)\Delta x_i$ for $\Delta_l U= U(a_1,...,x_i,...,a_n)-U(a_1,...,x_i-\Delta x_i,...,a_n)$;
- $\Delta_r U=U_{mr,i}(x)\Delta x_i$ for $\Delta_r U= U(a_1,...,x_i+\Delta x_i,...,a_n)-U(a_1,...,x_i,...,a_n)$;
- $\Delta_b U=U_{mb,i}(x)\Delta x_i$ for $\Delta_b U= \frac{U(a_1,...,x_i+\Delta x_i,...,a_n)-U(a_1,...,x_i-\Delta x_i,...,a_n)}{2}$

where $\Delta x_i>0$.

Before concluding this discussion let note that $U_{ml,i}$ in $(a_1,...,x_i,...,a_n)$ coincides with $U_{mr,i}$ in $(a_1,...,x_i-\Delta x_i,...,a_n)$ and also $U_{mr,i}$ in $(a_1,...,x_i,...,a_n)$ coincides with $U_{ml,i}$ in $(a_1,...,x_i+\Delta x_i,...,a_n)$. Also, from the above definition: $U_{mb.i}=\frac{U_{ml,i}+U_{mr,i}}{2}$ therefore: $\min\{U_{ml,i},U_{mr,i}\}\le U_{mb,i}\le\max\{U_{ml,i},U_{mr,i}\}$.

If the case of a differentiable utility of class C^1, we define **the marginal utility** in relation to the i-th good, while the consumption of other goods is constant, as:

$$U_{m,i}(x)=\frac{\partial U}{\partial x_i}(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)=$$

$$\lim_{\Delta x_i\to 0}\frac{U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x-\Delta x_{ii},a_{i+1},...,a_n)}{\Delta x_i}$$

therefore the differentiable marginal utility is the limit when of the discretized marginal utility when the variation of the good's consumption tends to 0.

The general approach of the utility function, requires it to be concave (the UC.1 axiom). But we have $d^2U=\sum_{i,j=1}^{n}\frac{\partial^2 U}{\partial x_i\partial x_j}dx_i dx_j=\frac{\partial^2 U}{\partial x_i^2}dx_i^2=\frac{\partial}{\partial x_i}\left(\frac{\partial U}{\partial x_i}\right)dx_i^2=\frac{\partial U_{m,i}}{\partial x_i}dx_i^2$ (caeteris paribus). The negatively defined character of d^2U implies $\frac{\partial U_{m,i}}{\partial x_i}<0$ therefore $U_{m,i}$ is decreasing caeteris paribus (Gossen's First Law).

Let now reconsider the situation of the discretized marginal utility. We saw that: $\Delta U=U_{m,i}(x)\Delta x_i$ caeteris paribus for each type of the marginal utility (but with

different meanings of ΔU). Considering a number of k units of good i consumed, we get (with abbreviated notation $U_i(j)=U_i(a_1,...,a_{i-1},j,a_{i+1},...,a_n)$ and analogously for $U_{m,i}$), successively, for the left marginal utility:

$$U_i(j+1)-U_i(j)=U_{ml,i}(j+1)\cdot 1 \ \forall j=\overline{0,k-1}$$

and after summing and reductions:

$$U_i(k)-U_i(0)=\sum_{j=1}^{k} U_{ml,i}(j)$$

We got that the total utility is the sum of discretized marginal utilities to the left. If it is one single good, we have $U_i(0)=0$ (the axiom U.3) thus:

$$U_i(k)=\sum_{j=1}^{k} U_{ml,i}(j)$$

We have obtained that the total utility corresponding to the consumption of k units of a good equals the sum of discretized marginal utilities to the left (for goods 1,...,k).

For the right marginal utility, we have:

$$U_i(j+1)-U_i(j)=U_{mr,i}(j)\cdot 1 \ \forall j=\overline{0,k-1}$$

and after summing and reductions:

$$U_i(k)-U_i(0)=\sum_{j=0}^{k-1} U_{mr,i}(j)$$

We got that the total utility is the sum of discretized marginal utilities to the right. If it is one single good, we have $U_i(0)=0$ (the axiom U.3) thus:

$$U_i(k)=\sum_{j=0}^{k-1} U_{mr,i}(j)$$

We have obtained that the total utility corresponding to the consumption of k units of a good equals the sum of discretized marginal utilities to the right (for goods 0,...,k-1 where the good 0 is formal in order to use the right utility).

For bilateral marginal utility, we have for a total number N of copies of good i:

$$\begin{cases} U_i(1)-U_i(0)=U_{mb,i}(0)\cdot 1 \\ \dfrac{U_i(j)-U_i(j-2)}{2}=U_{mb,i}(j-1)\cdot 1,\ j=\overline{2,N} \\ U_i(N)-U_i(N-1)=U_{mb,i}(N)\cdot 1 \end{cases}$$

After recurrence, it follows:

$$U_i(k) = U_i(k-2s) + 2\left(U_{mb,i}(k-2s+1) + ... + U_{mb,i}(k-1)\right) \ \forall s>0 \text{ such that } k-2s\geq 0$$

In particular, as $U_i(0)=0$ and $U_i(1)=U_{mb,i}(0)$ we have:

$$U_i(k) = 2\left(U_{mb,i}(1) + ... + U_{mb,i}(k-1)\right) \text{ for k=even}$$
$$U_i(k) = 2\left(U_{mb,i}(0) + U_{mb,i}(2) + ... + U_{mb,i}(k-1)\right) - U_{mb,i}(0) \text{ for k=odd}$$

Like a conclusion we have that the total utility corresponding to the consumption of k units of a good is equal to twice the sum of discretized bilateral marginal utilities of odd order less than k, and for k = odd with twice the sum of discretized bilateral marginal utilities of even order, less than k, minus the bilateral utility of the null good.

If the utility is differentiable, then:

$$U_i(k)=\int_0^k U_{m,i}(x)dx_i$$

and the corresponding marginal utility of the unit k of good i is:

$$U_{m,i}(k)=U_i(k)-U_i(k-1)=\int_{k-1}^{k} U_{m,i}(x)dx_i$$

In the general case of the variation in consumption of all existing goods, for k_1 units of good 1,...,k_n units of good n, we will consider first the simple way $\gamma:[0,1]\rightarrow \mathbf{R}^n$, $\gamma(t)=(tk_1,...,tk_n)$. This is nothing more than the large diagonal of the n-dimensional parallelepiped: $[0,k_1]\times...\times[0,k_n]$. Let also the differential form:

$$dU=\frac{\partial U}{\partial x_1}dx_1+...+\frac{\partial U}{\partial x_n}dx_n$$

that is continuous everywhere after the C^2 character of U. Along the path γ, the integral of dU is defined by:

$$\int_\gamma dU = \int_0^1 \left(\frac{\partial U}{\partial x_1}(\gamma_1(t),...,\gamma_n(t))\gamma'_1(t) + ... + \frac{\partial U}{\partial x_n}(\gamma_1(t),...,\gamma_n(t))\gamma'_n(t) \right) dt$$

where $\gamma_1,...,\gamma_n$ are the components of γ. The Leibniz-Newton's theorem for exact differential forms (forms with property $\exists U$ such that $\omega=dU$) states that: $\int_\gamma dU$ $=U(\gamma(1))-U(\gamma(0))$.

In the present case:

$$U(k_1,...,k_n)\text{-}U(0,...,0)=\int_0^1\left(\frac{\partial U}{\partial x_1}(k_1t,...,k_nt)k_1+...+\frac{\partial U}{\partial x_n}(k_1t,...,k_nt)k_n\right)dt=$$

$$k_1\int_0^1 U_{m,1}(k_1t,...,k_nt)dt+...+k_n\int_0^1 U_{m,n}(k_1t,...,k_nt)dt$$

Because U(0)=0, resulting the final formula:

$$U(k_1,...,k_n)=k_1\int_0^1 U_{m,1}(k_1t,...,k_nt)dt+...+k_n\int_0^1 U_{m,n}(k_1t,...,k_nt)dt$$

6. The Marginal Rate of Substitution

Let consider, first the case of two variable goods, the other remains fixed. Let the goods i and j with $i \neq j$. We define the space restriction of consumption: $G_{ij}=\{(x_1,..., x_n)\,|\,x_k=a_k=\text{const}, k=\overline{1,n}, k\neq i,j, x_i,x_j\in\mathbf{R}_+\}$ relative to the two goods where the others remain fixed. Also be: $D_{ij}=\{(x_i,x_j)\,|\,(x_1,...,x_n)\in G_{ij}\}$ - the consumption domain corresponding only to goods i and j.

We define: $u_{ij}:D_{ij}\to\mathbf{R}_+$ - the restriction of the utility function at goods i and j, i.e.:

$$u_{ij}(x_i,x_j)=U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_{j-1},x_j,a_{j+1},...,a_n)$$

The functions u_{ij} define a surface in $\mathbf{R}^3$ for any pair of goods (i,j).

We will call **partial marginal rate of substitution** between goods i and j, relative to G_{ij} (caeteris paribus), the variation of the amount of good j in order to substitute an amount of the good i in the hypothesis of utility conservation.

We will note in what follows:

$$RMS(i,j,G_{ij})=\frac{dx_j}{dx_i}$$

Since $u_{ij}(x_i,x_j)=\overline{u}=\text{const}$, we obtain by differentiation: $du_{ij}(x_i,x_j)=0$ i.e.: $\frac{\partial u_{ij}}{\partial x_i}dx_i+\frac{\partial u_{ij}}{\partial x_j}dx_j=0$ therefore: $\frac{dx_j}{dx_i}=-\frac{\frac{\partial u_{ij}}{\partial x_i}}{\frac{\partial u_{ij}}{\partial x_j}}=-\frac{\frac{\partial U}{\partial x_i}\Big|_{G_{ij}}}{\frac{\partial U}{\partial x_j}\Big|_{G_{ij}}}=-\frac{U_{m,i}\Big|_{G_{ij}}}{U_{m,j}\Big|_{G_{ij}}}$.

We can write: $RMS(i,j,G_{ij})=-\frac{U_{m,i}\Big|_{G_{ij}}}{U_{m,j}\Big|_{G_{ij}}}$ which is a function of x_i and x_j. In a fixed point $\overline{x}=\left(\overline{x}_1,...,\overline{x}_n\right)$ we have:

$$RMS(i,j,\bar{x})=-\frac{U_{m,i}(\bar{x})}{U_{m,j}(\bar{x})}$$

Let now consider the case when the consumption of all goods vary. Let therefore an arbitrary point $\bar{x}\in SC$ such that $U(\bar{x})=U_0=const$ and $U_{m,k}(\bar{x})\neq 0$, $k=\overline{1,n}$. Differentiating in $\bar{x}$ we obtain: $0=dU=\sum_{j=1}^{n}\frac{\partial U}{\partial x_j}dx_j$ therefore: $\frac{\partial U}{\partial x_i}+\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial U}{\partial x_j}\frac{dx_j}{dx_i}=0$ or, in terms of marginal utility: $U_{m,i}+\sum_{\substack{j=1\\ j\neq i}}^{n}U_{m,j}\frac{dx_j}{dx_i}=0$. If we note $\frac{dx_j}{dx_i}=y_j$, $j=\overline{1,n}$, $j\neq i$, we get: $U_{m,i}+\sum_{\substack{j=1\\ j\neq i}}^{n}U_{m,j}y_j=0$. With the aid of the partial substitution marginal rate introduced above, by dividing at $U_{m,i}$, we get:

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{y_j}{RMS(i,j,\bar{x})}=1$$

The above relationship is nothing but the equation of a hyperplane in $\mathbf{R}^{n-1}$ of coordinates $(y_1,...,\hat{y}_i,...,y_n)$ (the sign ^ means that that term is missing) that intersects the coordinate axes in $RMS(i,j,\bar{x})$. This hyperplane is the locus of consumption goods changes relative to a change in the consumption good "i" such that the utility remain constant.

For this reason, we will call the locus: **the marginal substitution hyperplane** between the good "i" and the other goods (note below $H_{mi,j}$).

In particular, for two goods, the marginal substitution hyperplane between the good i and the good j, of **R**, is reduced to: $\frac{y_j}{RMS(i,j,\bar{x})}=1$ where $y_j=\frac{dx_j}{dx_i}$. We have therefore $\frac{dx_j}{dx_i}=y_j=RMS(i,j,\bar{x})$ which is consistent with the definition of marginal rate of substitution.

We will define now the **overall marginal rate of substitution** between good i and the other as the opposite distance from the origin to the marginal substitution hyperplane, namely:

$$RMS(i,\bar{x})=-\frac{1}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{RMS^2(i,j,\bar{x})}}}=-\frac{U_{m,i}(\bar{x})}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}U_{m,j}^2(\bar{x})}}$$

We note that for the particular case of two goods, we get as above:

$$RMS(i,\bar{x})=-\frac{U_{m,i}(\bar{x})}{U_{m,j}(\bar{x})}$$

Considering now $v=(y_1,...,\hat{y}_i,...,y_n)\in H_{mi,j}$ we have: $\|v\|=\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}y_j^2}$ and from the Cauchy-Schwarz inequality:

$$\frac{\|v\|}{|RMS(i,\bar{x})|}=\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}y_j^2}\cdot\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{RMS^2(i,j,\bar{x})}}\geq\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{y_j}{RMS(i,j,\bar{x})}=1$$

that is: $\|v\|\geq|RMS(i,\bar{x})|$. By these results, the overall marginal rate of substitution is the minimum (in the meaning of norm) changes in consumption so that total utility remains unchanged.

Considering now the marginal substitution hyperplane: $\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{y_j}{RMS(i,j,\bar{x})}=1$ the equation of the normal line from the origin to it is:

$$\frac{y_1}{\frac{1}{RMS(i,1,\bar{x})}}=...=\frac{y_{i-1}}{\frac{1}{RMS(i,i-1,\bar{x})}}=\frac{y_{i+1}}{\frac{1}{RMS(i,i+1,\bar{x})}}=...=\frac{y_n}{\frac{1}{RMS(i,n,\bar{x})}}$$

therefore:

$$\begin{cases} y_1=\dfrac{\lambda}{RMS(i,1,\bar{x})}\\ \cdots\\ y_{i-1}=\dfrac{\lambda}{RMS(i,i-1,\bar{x})}\\ y_{i+1}=\dfrac{\lambda}{RMS(i,i+1,\bar{x})}\\ \cdots\\ y_n=\dfrac{\lambda}{RMS(i,n,\bar{x})} \end{cases},\ \lambda\in\mathbf{R}$$

The intersection of the normal line with the hyperplane, represents the coordinates of the point of minimum norm. We therefore have: $\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\lambda}{RMS^2(i,j,\bar{x})} = 1$ and $\lambda = \frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{RMS^2(i,j,\bar{x})}}$. The point of minimum norm has the coordinates:

$$RMS^2(i,\bar{x})\left(\frac{1}{RMS(i,1,\bar{x})},...,\frac{\hat{1}}{RMS(i,i,\bar{x})},...,\frac{1}{RMS(i,n,\bar{x})}\right) =$$
$$-\frac{U_{m,i}(\bar{x})}{\sum_{\substack{j=1\\ j\neq i}}^{n} U^2_{m,j}(\bar{x})}\left(U_{m,1}(\bar{x}),...,\hat{U}_{m,i}(\bar{x}),...,U_{m,n}(\bar{x})\right)$$

which norm is nothing else than $|RMS(i,\bar{x})|$.

The above coordinates of the point is no more than minimal vector (in the meaning of norm) of the consumption changes such that the utility remain unchanged. We will say briefly that this is **the minimal vector** of the i-th good **substitution**.

For the discrete case, we will define the marginal rate of substitution between goods "i" and "j", caeteris paribus, the quantity of good "j" required for replacement a unit of "i" in the situation of the utility conservation.
Let us recall that, in the case of left discretized:

$$U_{ml,i}(\bar{x}) = \frac{U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i,\bar{x}_{i+1},...,\bar{x}_n) - U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i - \Delta x_i,\bar{x}_{i+1},...,\bar{x}_n)}{\Delta x_i} \quad \forall i=\overline{1,n}$$

from where:

$$U_{ml,i}(\bar{x})\,\Delta x_i + U_{ml,j}(\bar{x})\,\Delta x_j =$$
$$2U(\bar{x}_1,...,\bar{x}_n) - U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i - \Delta x_i,\bar{x}_{i+1},...,\bar{x}_n) - U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j - \Delta x_j,\bar{x}_{j+1},...,\bar{x}_n)$$

For very small variations of x_i and x_j, respectively, therefore $\Delta x_i \approx 0$, $\Delta x_j \approx 0$ we get:

$$U_{ml,i}(\bar{x})\,\Delta x_i + U_{ml,j}(\bar{x})\,\Delta x_j \approx 0$$

or:

$$U_{ml,i}(\bar{x})\,\Delta x_i \approx - U_{ml,j}(\bar{x})\,\Delta x_j$$

We have therefore the **left partial marginal rate of substitution** between goods "i" and "j":

$$RMS_l(i,j,\bar{x}) = \frac{\Delta x_j}{\Delta x_i} \approx -\frac{U_{ml,i}(\bar{x})}{U_{ml,j}(\bar{x})}$$

Also in the case of right discretized:

$$U_{mr,i}(\bar{x}) = \frac{U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i+\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n) - U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i,\bar{x}_{i+1},...,\bar{x}_n)}{\Delta x_i} \quad \forall i=\overline{1,n}$$

from where:

$$U_{mr,i}(\bar{x})\,\Delta x_i + U_{mr,j}(\bar{x})\,\Delta x_j =$$
$$U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i+\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n) + U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j+\Delta x_j,\bar{x}_{j+1},...,\bar{x}_n) - 2U(\bar{x}_1,...,\bar{x}_n)$$

At very small variations of x_i and x_j:

$$U_{mr,i}(\bar{x})\,\Delta x_i + U_{mr,j}(\bar{x})\,\Delta x_j \approx 0$$

therefore:

$$U_{mr,i}(\bar{x})\,\Delta x_i \approx -U_{mr,j}(\bar{x})\,\Delta x_j$$

We can coclude that the **right partial marginal rate of substitution** between goods "i" and "j" is:

$$RMS_r(i,j,\bar{x}) = \frac{\Delta x_j}{\Delta x_i} \approx -\frac{U_{mr,i}(\bar{x})}{U_{mr,j}(\bar{x})}$$

In the bilateral discretized case:

$$U_{mb,i}(\bar{x}) = \frac{U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i+\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n) - U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i-\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n)}{2\Delta x_i}$$
$$\forall i=\overline{1,n}$$

therefore:

$$U_{mb,i}(\bar{x})\,\Delta x_i + U_{mb,j}(\bar{x})\,\Delta x_j =$$

$$\frac{U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i+\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n) - U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i-\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n)}{2} +$$
$$\frac{U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j+\Delta x_j,\bar{x}_{j+1},...,\bar{x}_n) - U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j-\Delta x_j,\bar{x}_{j+1},...,\bar{x}_n)}{2}$$

or:

$$2\left(U_{mb,i}(\bar{x})\Delta x_i + U_{mb,j}(\bar{x})\Delta x_j\right) =$$
$$U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i+\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n) + U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j+\Delta x_j,\bar{x}_{j+1},...,\bar{x}_n)$$
$$-U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i-\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n) - U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j-\Delta x_j,\bar{x}_{j+1},...,\bar{x}_n)$$

At very small variations of x_i and x_j:

$$U_{mb,i}(\bar{x})\,\Delta x_i + U_{mb,j}(\bar{x})\,\Delta x_j \approx 0$$

or, equivalent:

$$U_{mb,i}(\bar{x})\,\Delta x_i \approx - U_{mb,j}(\bar{x})\,\Delta x_j$$

We will define therefore the **bilateral partial marginal rate of substitution** between goods "i" and "j" is:

$$RMS_b(i,j,\bar{x}) = \frac{\Delta x_j}{\Delta x_i} \approx -\frac{U_{mb,i}(\bar{x})}{U_{mb,j}(\bar{x})}$$

Let us note now for simplicity:

$$\alpha_k = \frac{U(\bar{x}_1,...,\bar{x}_{k-1},\bar{x}_k + \Delta x_k,\bar{x}_{k+1},...,\bar{x}_n)}{\Delta x_k}$$

$$\beta_k = \frac{U(\bar{x}_1,...,\bar{x}_{k-1},\bar{x}_k,\bar{x}_{k+1},...,\bar{x}_n)}{\Delta x_k} \qquad \forall k = \overline{1,n}$$

$$\gamma_k = \frac{U(\bar{x}_1,...,\bar{x}_{k-1},\bar{x}_k - \Delta x_k,\bar{x}_{k+1},...,\bar{x}_n)}{\Delta x_k}$$

With these notations, we get then:

$$U_{mb,k}(\bar{x}) = \frac{\alpha_k - \gamma_k}{2} = \frac{(\alpha_k - \beta_k) + (\beta_k - \gamma_k)}{2} = \frac{U_{ml,k}(\bar{x}) + U_{mr,k}(\bar{x})}{2} \quad \forall k = \overline{1,n}$$

For the three cases above, we therefore:

$$RMS_b(i,j,\bar{x}) = -\frac{U_{mb,i}(\bar{x})}{U_{mb,j}(\bar{x})} = -\frac{U_{ml,i}(\bar{x}) + U_{mr,i}(\bar{x})}{U_{ml,j}(\bar{x}) + U_{mr,j}(\bar{x})}$$

After this formula, we get:

$$(RMS_b(i,j,\bar{x}) - RMS_l(i,j,\bar{x}))(RMS_b(i,j,\bar{x}) - RMS_r(i,j,\bar{x})) =$$

$$-\frac{(U_{ml,i}(\bar{x})U_{mr,j}(\bar{x}) - U_{ml,j}(\bar{x})U_{mr,i}(\bar{x}))^2}{U_{ml,j}(\bar{x})U_{mr,j}(\bar{x})(U_{ml,j}(\bar{x}) + U_{mr,j}(\bar{x}))^2} < 0$$

by virtue of the fact that marginal utilities are positive.

Following this result, we get a not surprising result: $RMS_b(i,j,\bar{x})$ is situated between $RMS_l(i,j,\bar{x})$ and $RMS_r(i,j,\bar{x})$ so it is the best approximation for the partial marginal rate of substitution.

A better approximation for the partial marginal rates of substitution can be taken given the fact that at a decrease in consumption of a product, the left marginal utility of consumption meaning this direction, while the right marginal utility is much more

useful if the consumption growth. Therefore, to estimate the changes in consumer in the direction of the decreasing for the good "i" and, obviosly for the increasing in the case of "j", we will compute **the adjusted decreasing partial marginal rate of substitution**:

$$RMS_{aj,dec}(i,j,\bar{x})=-\frac{U_{ml,i}(\bar{x})}{U_{mr,j}(\bar{x})}$$

and analogously, to estimate changes in consumer in the direction of the increasing for the good "i" and for the decreasing in the case of "j", we will compute **the adjusted increasing partial marginal rate of substitution**:

$$RMS_{aj,inc}(i,j,\bar{x})=-\frac{U_{mr,i}(\bar{x})}{U_{ml,j}(\bar{x})}$$

For arbitrary evolutions in consumption, we will use or (for simplicity) **the bilateral partial marginal rate of substitution**:

$$RMS_{b}(i,j,\bar{x})=-\frac{U_{mb,i}(\bar{x})}{U_{mb,j}(\bar{x})}$$

which is between the two adjusted rates because:

$$(RMS_b(i,j,\bar{x})- RMS_{aj,dec}(i,j,\bar{x}))(RMS_b(i,j,\bar{x})- RMS_{aj,inc}(i,j,\bar{x}))=$$
$$-\frac{\left(U_{mr,j}(\bar{x})U_{mr,i}(\bar{x})-U_{ml,i}(\bar{x})U_{ml,j}(\bar{x})\right)^2}{\left(U_{ml,j}(\bar{x})+U_{mr,j}(\bar{x})\right)^2 U_{mr,j}(\bar{x})U_{ml,j}(\bar{x})}<0$$

Similarly, for the discretized case, the overall marginal rate of substitution will be:

$$RMS_{l}(i,\bar{x})=\frac{1}{\Delta x_i\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{(\Delta x_j)^2}}}\approx-\frac{U_{ml,i}(\bar{x})}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}U^2_{ml,j}(\bar{x})}}$$

$$RMS_{r}(i,\bar{x})=\frac{1}{\Delta x_i\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{(\Delta x_j)^2}}}\approx-\frac{U_{mr,i}(\bar{x})}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}U^2_{mr,j}(\bar{x})}}$$

$$RMS_b(i,\bar{x})=\frac{1}{\Delta x_i\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{(\Delta x_j)^2}}}\approx-\frac{U_{mb,i}(\bar{x})}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}U^2_{mb,j}(\bar{x})}}$$

7. Exemples of Preferences

7.1. Perfectly Substitutable Goods

We say that n goods are perfectly substitutable if the utility function is linear, i.e.:

$$U(x_1,...,x_n)=a_1x_1+...+a_nx_n,\ a_i>0,\ i=\overline{1,n}$$

In this case, we have: $U_{m,i}=\frac{\partial U}{\partial x_i}=a_i,\ i=\overline{1,n}$ and the partial substitution marginal rate is:

$$RMS(i,j,\bar{x})=-\frac{a_i}{a_j}$$

whereas the overall marginal rate of substitution is:

$$RMS(i,\bar{x})=-\frac{a_i}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}a_j^2}}$$

Also the marginal substitution hyperplane between good “i” and the other goods has the equation:

$$\sum_{\substack{j=1\\ j\neq i}}^{n}a_jy_j+a_i=0$$

We can see in this case that both the marginal rate of substitution and the overall are constant. This implies that whatever is the level of consumption, the substitutability between any two goods, caeteris paribus, has the same factor. The minimal vector of the i-th good substitution is:

$$-\frac{a_i}{\sum_{\substack{j=1\\ j\neq i}}^{n}a_j^2}(a_1,...,\hat{a}_i,...,a_n)$$

7.2. Independent Goods from the Utility Point of View

We will say that n goods are independent from utility the point of view if the utility function is:

$U(x_1,...,x_n)=f_1(x_1)+...+f_n(x_n)$, with $f_i\in C^2(0,\infty)$, $f_i''\le 0$, $i=\overline{1,n}$ and $f_1(0)+...+f_n(0)=0$

In this case we have:

$$U_{m,i}=\frac{\partial U}{\partial x_i}=f_i'\text{, } i=\overline{1,n}\text{, } \frac{\partial^2 U}{\partial x_i^2}=f_i''\text{, } i=\overline{1,n}\text{, } \frac{\partial^2 U}{\partial x_i\partial x_j}=0\ \ \forall i\neq j$$

Because $d^2U=\sum_{i,j=1}^{n} f_i''dx_i^2$ follows that U is concave.

Before proceeding further, let note that in the case of the linearity of functions f_i ($f_i(x_i)=a_ix_i$, $a_i>0$) the goods become perfectly substitutable.

The partial substitution marginal rate is: $RMS(i,j,\bar{x})=-\frac{f_i'(\bar{x}_i)}{f_j'(\bar{x}_j)}$ and the overall marginal rate of substitution: $RMS(i,\bar{x})=-\frac{f_i'(\bar{x}_i)}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n} f_j'^2(\bar{x}_j)}}$.

Also the marginal substitution hyperplane between good "i" and the other goods has the equation:

$$\sum_{\substack{j=1\\ j\neq i}}^{n} f_j'(\bar{x}_j)y_j+f_i'(\bar{x}_i)=0$$

The minimal vector of the i-th good substitution is therefore:

$$-\frac{f_i'(\bar{x}_i)}{\sum_{\substack{j=1\\ j\neq i}}^{n} f_j'^2(\bar{x}_j)}\left(f_1'(\bar{x}_1),...,\hat{f}_i'(\bar{x}_i),...,f_n'(\bar{x}_n)\right)$$

7.3. Separable Goods from the Utility Point of View

We will say that n goods are separable from the utility point of view if the utility function is:

$U(x_1,...,x_n)=f_1(x_1)\cdot...\cdot f_n(x_n)$, cu $f_i\in C^2(0,\infty)$, $f_i(x)>0$ $\forall x>0$, $f_i''\le 0$, $i=\overline{1,n}$, $f_1(0)\cdot...\cdot f_n(0)=0$

and the quadratic form:

$$H=\sum_{i=1}^{n}\frac{f_i''}{f_i}\xi_i^2+\sum_{i,j=1}^{n}\frac{f_i'f_j'}{f_if_j}\xi_i\xi_j$$

is negatively defined.
In this case, we have:

$$U_{m,i}=\frac{\partial U}{\partial x_i}=f_1...f_i'...f_n=U\frac{f_i'}{f_i},\ i=\overline{1,n},\ \frac{\partial^2 U}{\partial x_i^2}=f_1...f_i''...f_n=U\frac{f_i''}{f_i},\ i=\overline{1,n},$$

$$\frac{\partial^2 U}{\partial x_i\partial x_j}=f_1...f_i'...f_j'...f_n=U\frac{f_i'f_j'}{f_if_j}\ \ \forall i\neq j$$

From the fact that H is a negative defined quadratic form, it follows that U is concave.
The partial substitution marginal rate is: $RMS(i,j,\bar{x})=-\frac{f_i'(\bar{x}_i)f_j(\bar{x}_j)}{f_j'(\bar{x}_j)f_i(\bar{x}_i)}$ and the overall marginal rate of substitution is: $RMS(i,\bar{x})=-\frac{f_i'(\bar{x}_i)}{f_i(\bar{x}_i)\sqrt{\sum_{\substack{j=1\\j\neq i}}^{n}\frac{f_j'^2(\bar{x}_j)}{f_j^2(\bar{x}_j)}}}$.
Also the marginal substitution hyperplane between the i-th good and the others has the equation:

$$\sum_{\substack{j=1\\j\neq i}}^{n}\frac{f_j'(\bar{x}_j)}{f_j(\bar{x}_j)}y_j+\frac{f_i'(\bar{x}_i)}{f_i(\bar{x}_i)}=0$$

and the minimal vector of goods substitution:

$$-\frac{f_i'(\bar{x}_i)}{f_i(\bar{x}_i)\sum_{\substack{j=1\\j\neq i}}^{n}\frac{f_j'^2(\bar{x}_j)}{f_j^2(\bar{x}_j)}}\left(\frac{f_1'(\bar{x}_1)}{f_1(\bar{x}_1)},...,\frac{\hat{f}_i'(\bar{x}_i)}{f_i(\bar{x}_i)},...,\frac{f_n'(\bar{x}_n)}{f_n(\bar{x}_n)}\right)$$

Let consider, as a particular example, the Cobb-Douglas function:

$$U(x_1,x_2,...,x_n)=x_1^{\alpha_1}...x_n^{\alpha_n}\ \text{cu}\ \alpha_i>0,\ \sum_{i=1}^{n}\alpha_i\leq 1$$

Computing the marginal partial rate of substitution, we get: $RMS(i,j,\bar{x})=-\frac{\alpha_i\bar{x}_j}{\alpha_j\bar{x}_i}$
and the overall marginal rate of substitution: $RMS(i,\bar{x})=-\frac{\alpha_i}{\bar{x}_i\sqrt{\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\alpha_j^2}{\bar{x}_j^2}}}$. Also the marginal substitution hyperplane between the i-th good and the others has the equation:

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\alpha_j}{\bar{x}_j} y_j + \frac{\alpha_i}{\bar{x}_i} = 0$$

8. Conclusions

The onset of the notions of preference in selecting baskets of goods and the utility on the other hand impose a number of precautions both conceptual and technical. Even if such an axiomatic constraint will lead to "loss" of some important cases, the axiomatisation gives, on the one hand, rigor to the theory and, on the other hand, generates new situations by using norms more or less exotic (1-norm, ∞-norm, p-norms).

On the other hand, the analysis of the concurrent consumption of n goods variance, decontrols the theory from "caeteris paribus" constraints, getting interesting conclusions and making a first step towards the overall analysis of the microeconomic phenomenon. The n-dimensional approach to the basic phenomena, even if it is based on a number of notions of n-dimensional Euclidean geometry or, in the overall treatment of the utility function, a series of differential geometry results, adds generality and more accurate simulation of microeconomic reality.

Also the treatment of the marginal utility to the left, right or bilateral as well as marginal rates of substitution of different types, introduced above, allows enrichment practical conclusions, eliminating the classical mono-directional variations like the discretized derivative derived as the average of discretized left and right derivatives gives more precise information on the behavior of a function.

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Contribution of Financial Development in Electricity-Growth Nexus in Pakistan

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Abstract: The perseverance of the paper is to scrutinize the causal connection within economic uplift and electricity consumption in Pakistan. There is a need to incorporate a third variable as a financial development in traditional electricity-growth model by creating a tri-variate causality framework; so the problem of omitted variables bias in this nexus has been eradicated. The application of co-integration along with vector error correction model in our study validates a uni-variate an extensive rise in causality in succession from electricity consumption towards financial development and economic development. These results spring strength to argument according to that Pakistan is a state which depends on energy. Every set back in the power supply retards the development process and in consequence the economy moves towards evil of backwardness. In addition, results show that financial development as well as economic development in Pakistan raises the electricity consumption, and the outcomes are applicable to all regardless of the long run causality observed in Pakistan. Hence the study commends geared headed for the expansion of power sector through FDI influx and Credit plans in order to handle the rising electricity call utilized by rapid industrialization and growth in the economy.

Key Words: electricity consumption; foreign direct investment; gross domestic product; financial development

JEL Classification: C33; O52; Q43

1. Introduction

The ties between electricity consumption, financial development and economic development have recently enduring to be claimed in energy-growth literature. This literature can be alienated into three streaks. In this paper the primary line of exploration emphases on the nexus amid economic evolution and energy consumption. This nexus advocates that economic progression and energy consumption may be mutually resolute, because higher economic progression necessitates more energy consumption. Similarly, more competent use of energy needs a upper level of economic growth. Therefore, the trend of interconnection may not be determined prior. Meanwhile the innovator Kraft and Kraft (1978), the

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Granger causality test method has become a prevalent tool for reviewing the affiliation between economic growth and energy consumption in unalike countries (Ghosh, 2010; Ozturk & Acaravci, 2010) and this takes to four experiment able hypotheses; (1) a causal relationship from energy consumption towards economic evolution, (2) a causal relationship from economic evolution towards energy, (3) a feed-back correlation within energy and GDP, and (4) no causal association within energy and economic evolution (neutrality).

Second area of this research is to scan the interrelationship within financial development by considering FDI and Economic evolution. It was considered that the FDI through capital accumulation plays a dominant role in uplift of the economy. However it's has been widely argued in the literature and has been considered as one of the fundamental concepts of economics. A number of studies have also been observed in the literature discussing the role of FDI in the capital formation, later used as a tool for evaluation of economic growth (Omri and Kahouli, 2013). A number of studies like Bloomstrom et al. (1996); Aitken et al. (1997); Lee and Chang (2009) defines the direction of causality from FDI towards economic development. Furthermore, studies like Hsiao and Shen (2003) also stress upon the importance of FDI in economic uplift by validating the economy's dependence upon this factor. In contrast few studies like Nguyen and Nguyen (2007); Anwar and Nguyen (2010) define the two way associationship between FDI and economic evolution by considering different nations of the world. Apart from such discussion, studies in this strand of literature also lead towards different results and it has been concluded that such difference is due to the country specific conditions (Zhang, 2001).

The third dimension of this research has scrutinized the associationship between financial development (FDI) and electricity consumption. In this regard, tang (2009) opt a stance that the influx of FDI encourage the demand of electricity through expansion of transportation, industrialization and manufacturing sector development. However this area of researcher is not well crowded and still has a window of opportunity as open to accept the latest studies. In this regard, studies like Mielnik and Goldemberg (2000); Sadorsky (2010) claims the positive associationship between electricity consumption and FDI influx in the economy. They state that the involvement of FDI in the economy raises the competition as a consequence of which business became cheaper and also ensure the easy availability of capital, which can be casted to expand the production sector as a consequence of which demand for power resources will be improved and move towards new levels. A main limitation of different results has also been observed in this area of research as we find in the previous lines of research.

So, by keeping the results of such previous literature under consideration, it has been concluded that the higher level of development in the economy can be attained through the addition of power resources to the economy, which not only raise the

production sector of the economy, also raises the involvement of foreign interest towards the economy. In this situation, it is worthwhile to empirically investigate the associationship between economic development, electricity consumption and financial evolution. The present study is unique from the existing literature in the following manners. Compared to the previous literature, this study does not restrict itself towards traditional bi-variate model of relationship, but add financial development as a third variable in the traditional bi-variate model in order to eliminate the omitted variable problem from the relationship. This study further employs principal component analysis (PCA) approach for the measurement of financial development indicators, which helps us to find the suitable indicator without hurting the data. As far as we know, this study became first in itself, which applies the concept of PCA in energy growth nexus in Pakistan and hence represents our contribution towards the existing strand of knowledge.

In terms of objectivity and practical application, this study will help the government and other policy-making organizations to design suitable policies to grab the evil of power shortage, which has become one of the challenging tasks for the Government of Pakistan. The results will define the nature as well as direction of the relationship, which will enable the government and other policy-making organizations to take relevant steps regarding promotion of trade, liberalization of the electricity sector or financial sector. This will help the economy to come out from this state of the economic downturn and smoothen its way towards economic development.

The pattern of this study is as follows: Section 1 gives an introduction of the topic, section 2 define the review of the literature. Section 3 represents the theoretical model, data as well as a methodology of the study. However, interpretation of results and conclusion are defined in section 4 and 5 respectively.

2. Review of Literature

Enormous number of studies has been conducted with a view to relate financial sector development with economic evolution in different areas of the world (Schumpeter, 1911; Shaw, 1973; Mckinnon, 1973; Arestis & Demetriades, 1997; Shahbaz at el, 2013 and Ozturk & Acaravci, 2013). However, these studies contribute to the literature by providing different dimensions for measurement. Among these dimensions, two dimensions named as factor accumulation and factor productivity are defined by Sadorsky (2010). Both of these approaches raise the efficiency of financial systems by reducing the problems of information asymmetries, which increase the contribution of these studies in the economic development of the nations.

A number of studies (Claessens & Leaven, 2004; Karanfil, 2009; Sadorsky, 2010; Sadorsky, 2011) are of the view that energy-growth relationship is not justified through simple bivariate model; but there is another important element whose presence in relationship is not ignorable, and that is financial development. Similarly, Shahbaz and Lean (2012) conducted a study in Tunisia and found a positive bi-directional relationship of industrialization with financial development **(FD↔IND),** financial development with energy consumption **(FD↔EC)** and Industrialization with energy consumption **(IND↔EC)**. Sadorsky (2010) uses the GMM approach to measuring the financial development impression on economic progress of 22 developing nations. He uses stock market approach to measure the financial development and found significant results of the existence of the relationship between economic uplift and financial sector development in developing nations (Sadorsky, 2010). Similarly in another study he uses the same concept of identification of the association between electricity consumption and financial development in panel of 9 frontier nations of Central and Eastern Europe by employing same GMM technique (Sadorsky, 2011). In this paper, he used four banking and three stock market indicators for the measurement of financial development and concluded that each of the banking variables is statistically significant in this case. These results validate his previous findings of presence of a positive relationship between electricity consumption and financial sector development of the country (Sadorsky, 2010).

Apart from studies of Sadorsky, considerable work done by Yandan and Lijun (2009) in Guangdong for the purpose of primary electricity consumption relationship with the financial development of the state. Authors employ Granger causality and cointegration analysis to investigate the existence of causality in either short or long term in Guangdong. Results indicate the presence of stability in cointegration on a short term, while variations in long-term relationship. Furthermore, the study also states that financial development in Guangdong is a reason of rise in its primary electric consumption, which is an evidence of one-way causal relationship in this study **(FD→EC)** (Yandan & Lijun, 2009).

In energy growth nexus, significant work is done on the determination of energy consumption and economic development (Karanfil, 2009; Barleet and Gounder, 2010; Ozturk & Acravci, 2010; Narayan et al, 2010; Saatci & Dumrul, 2013). All these studies resulted with a stable form of relationship between these variables. Among these studies, Hou (2009) also address this traditional relationship between EC and EG in China with the help of ADF, Cointegration and Hsiao's ranger causality in China. The results of his study show an existence of a bidirectional relationship in which energy consumption along with economic growth Granger cause each other **(EC↔EG)** (Hou, 2009).

Islam et al. (2013) conducted a study on Malaysian economy and used a vector error correction technique and demonstrates the existence of bidirectional causality between financial development and electricity consumption **(FD↔EC)**. The study of a similar nature was conducted by Shahbaz et al. (2012) in Tunisian economy, and shows results in the form of relationship between energy consumption, financial industrialization, as well as financial development. Similarly Salman and Atya (2014) carried a study on North African nations covering period of 1980 to 2010 for validation of causality between financial development and economic growth, and check its impact on electricity consumption. They applied Granger causality as well as error correction model and demonstrate the existence of positive causality in energy finance relationship in Algeria as well as Tunisia, while negative correlation in Egypt due to very minimal amount of subsidies in power sector.

In case of Pakistan, Hye and Riaz (2008) conducted a study for governing the dimension of causality between ED (economic development) and EC (electricity consumption) through application of augmented form of Granger causality as well as bound testing approach of cointegration. Results of this study indicate the presence of short-term bidirectional causality between electricity consumption and economic development **(EC↔ED)**. On the other side of the picture, the existence of unidirectional causality indicates that the economic development is the main reason of rise in electricity consumption in the economy. This is because the involvement of big tickets in economy raises the consumption and to meet this consumption; demand of oil increases, which in turn increase the electricity prices.

Another study conducted by Shahbaz and Fetidun (2011) also supports the results of Hye and Riaz (2008) about the presence of long-run unidirectional causality from economic growth towards electricity consumption **(EG→EC)**. The results of the study also show the existence of the relationship between employment level of the country and electricity consumption, which in a broader sense supports the argument of Sadorsky (2010, 2011) about big tickets. On the other hand Aqeel and Butt (2001) contradict the views of energy growth nexus, and show that the causality exists, but in the opposite direction as defined by previous studies (Hye and Riaz, 2008 and Shahbaz and Feridun, 2011). Javid et al. (2013) identifies the presence of unidirectional long-term causality, in which electricity consumption causes the rise in the per capita GDP. The reason for this causlaity is the direct relationship between electricity consumption and production process, which will raise the employment as well as exports of the country resulting in a rise per capita GDP. They also showed the deviations in the short term by defining electricity consumption as a restraining factor of economic growth in Pakistan. A significant portion of economy's earning derives from production sector and any shock or shortfall in electricity results in a reduction in production level, which in turn reduces the exports and cause adverse effects on economic growth. Kakar et al. (2011) aimed to identify the nature of the

relationship between economic development, electricity consumption and financial development variables. Results of his study show compliance with other studies through the presence of long-run unidirectional causality from financial development to electricity consumption, while contradiction through inexistence of the short-term relationship. However, direction of causality is found from EG to EC and from FD to EC.

Apart from these studies, this study uses principal component analysis in accordance with Tang and Tan (2014) to measure the financial development. Along with PCA, cointegration and regression analysis is also adopted to define its impact of FD on FDI inflow as well as electricity consumption and economic growth either in short or long-term.

3. Theoretical Background

In this study, a framework based upon the theory of consumer behavior is employed. This theory like any other theory of finance is based on the concept of rationality. These theories say that despite different values and perceptions in the society, one thing became common among all the customers, and is the maximization of benefit within limited resources. However, there are different items that can affect the consumption pattern of a person and define the consumer consumption pattern. In this study, we use a portion of such factors to define a model to measure electricity consumption.

$$\ln EC_t = a_0 + a_1 \ln GDP_t + a_2 \ln FDI_t + a_3 \ln FD_t + \varepsilon_t$$

EC stands for electricity consumption measured in kilowatt-hour; FDI represents the inflow of FDI measured in local currency; GDP includes the real per capita GDP in resident currency and FD becomes an indicator of financial development.

3.1. Measurement of Financial Development

Karanfil (2009) consider financial development as one of those other variables that affect not only the electricity demand, but also the relationship between EC and ED. However, this concept of financial development cannot be measured through any single variable, as it contains a number of similar variables that are extensively used in existing literature for measurement of financial development. This study uses banking sector approach in accordance with Ang (2009); Ang and Mckibbon (2007); Sadorsky (2011); Tang and Tan (2014) and used Domestic credit provided by banking sector, financial system deposit, broad money (M2), Liquid Liabilities (M3), Domestic Credit to Private Sector (PRI) to measure financial development in Pakistan. All these proxies of financial development are in percentage form of GDP.

Banking sector approach is used because it helps in the determination of financial depth of the economy. On the other hand, depth defines size of the financial sector with respect to the economy indicators. The more the depth, higher will be the financial sector, and greater will be the industrialization in the economy. On the other hand, higher the industrialization, the greater will be electricity consumption, because of the rise in per capita income that stimulates not only the demand of big tickets, but also the economic growth of the economy.

Traditionally M2 and M3 are used for determination of financial development, but these variables are related to one another and show the flow of funds instead their channelization from depositors to investors in the economy (Ang & Mckibbin, 2007). In this regard, credit provided to a private channel is considered as suitable because it shows the efficiency of private sector to channelize funds in an efficient manner as compared to public sector. However, all of these variables of FD used in this study are closely related to each other, so their inclusion in a single equation raises a problem of multi-collinearity. In order to eliminate the risk of such problem, this study adopts a technique named as Principal Component analysis for measurement of financial development. PCA is a method used for the reduction of dimensional problem in the data (Coban & Topcu, 2013). PCA reduces the number of variables by including their information in a group of principle components. Among these groups, the first component shows the maximum variance, and if more than one component exists; then the amount of variance is distributed among components. Each principle shows the variance through its Eigenvalues that display the percentage of total data explained by the component.

3.2. Data and Methodology

This study focuses on the annual time series data of 1975-2011, collected from World Bank's published and unpublished resources (WDI). All the variables selected for this study represent time series data, so there is a possibility of the presence of trends in the data. In order to resolve this problem, we procured the log of all the variables used in this study. In order to further remove the problem of trends in data, we calculate the unit root via Augmented Dickey Filler test, so that all the variables became stationary at any single period.

3.3. Unit Root analysis

In any time series data, there is always a possibility of the presence of trends, and it is essential to integrate the data at any single level of integration for getting fruitful results. However with this view, we also apply ADF (augmented Dickey filler) test in this study to check the presence of integration or stationarity in the data at any certain level by using the following equation.

$$\Delta Y_t = \alpha + \delta Y_{t-1} + U_t$$

3.4. Cointegration analysis

In order to judge the presence of any level of cointegration between EC, FDI, GDP and FD variables, this study employs the Johansen method of cointegration in accordance with the study of Tang and Tan (2014). However, the Johansen cointegration approach can be calculated by adopting the vector autoregression (VAR) model with the following estimator.

$$\Delta Z_t = \emptyset D_t + \prod Z_{t-1} + \sum_{j=1}^{k-1} \Gamma \Delta Z_{t-j} + \upsilon_t$$

The Δ sign indicates the difference operator; Z represents the vector of all the endogenous variables that includes the log of EC, FDI, GDP and FD. $\prod$ defines the matrix containing long-term information of all the variables while Γ represents the matrix of all the unknown parameters. For taking any decision regarding the presence of cointegration or association among the stated variables, this study considered P values of Trace statistics and Maximum Eigenvalue tests. If the P-value is less than 5%, Null hypothesis of no cointegration among variables is rejected while alternative hypothesis of the presence of cointegration is accepted and vice versa.

3.5. Vector Error Correction Term

The direction of causality can easily calculated through the application of traditional causality approach defined by Engle and Granger (1987). Such application through standard VAR model is misleading because of the stationarity of all the variables at same first order I(1). In this particular situation, causality can be determined through the application of error correction model.

$$\Delta \ln EC_t = \alpha_1 + \sum_{i=1}^{p} \beta_i \Delta \ln EC_{t-i} + \sum_{i=1}^{p} \vartheta_i \Delta \ln GDP_{t-i} + \sum_{i=1}^{p} \kappa_i \Delta \ln FD_{t-i} + \sum_{i=1}^{p} \theta_i \Delta \ln FDI_{t-i} + + \psi_1 ECT_{t-1} + \upsilon_{1t} + \varsigma 1t$$

$$\Delta \ln GDP_t = \alpha_2 + \sum_{i=1}^{p} \vartheta_i \Delta \ln GDP_{t-i} + \sum_{i=1}^{p} \beta_i \Delta \ln EC_{t-i} + \sum_{i=1}^{p} \kappa_i \Delta \ln FD_{t-i} + \sum_{i=1}^{p} \theta_i \Delta \ln FDI_{t-i} + \psi_2 ECT_{t-1} + \upsilon_{2t} + \varsigma 2t$$

$$\Delta \ln FD_t = \alpha_3 + \sum_{i=1}^{p} \kappa_i \Delta \ln FD_{t-i} + \sum_{i=1}^{p} \beta_i \Delta \ln EC_{t-i} + \sum_{i=1}^{p} \vartheta_i \Delta \ln GDP_{t-i} + \sum_{i=1}^{p} \theta_i \Delta \ln FDI_{t-i} + \psi_3 ECT_{t-1} + \upsilon_{3t} + \varsigma 3t$$

$$\Delta \ln FDI_t = \alpha_4 + \sum_{i=1}^{p} \theta_i \Delta \ln FDI_{t-i} + \sum_{i=1}^{p} \beta_i \Delta \ln EC_{t-i} + \sum_{i=1}^{p} \vartheta_i \Delta \ln GDP_{t-i} + \sum_{i=1}^{p} \kappa_i \Delta \ln FD_{t-i} + \psi_4 ECT_{t-1} + \upsilon_{4t} + \varsigma 4t$$

In these above mentioned equations elements of both long and short term are involved. Δ denotes the first difference while *ln* denotes the natural logarithm of a variable, while ςit represents the value of white noise and normally distributed residuals. On the other hand, residuals are denoted by υ_{it} and they are considered as normally distributed. As we mentioned earlier ECT_{t-1} represents lagged error term, which we obtained through the application of cointegration. In contrast, short-term direction causality is determined through application of Wald test on the both lagged error term and lagged value of exogenous variable.

4. Empirical Findings

4.1. Results of Principal Component Analysis

Principal component analysis is labeled as a tool applied to reduce the dimensionality of the variables by addressing the variance in form of principal components. The study employed principle component analysis for identification of the suitable measure of financial development. In this regard, four parameters of the banking sector are used to measure the depth of the financial sector in Pakistan. The results of PCA are shown in Table 1.

Table 2. Principal Component Analysis

Principal Component	Eigenvalues	% of Variance	Cumulative %
1	3.462113	0.6924	0.6924
2	1.064093	0.2128	0.9052
3	0.339304	0.0679	0.9731
4	0.107042	0.0214	0.9945
5	0.027448	0.0055	1.0000

Variable	Principal Component 1	Principal Component 2	Principal Component 3	Principal Component 4	Principal Component 5
LOGM3GDP	0.521747	0.179519	0.107059	0.085796	-0.822636
LOGM2GDP	0.512555	-0.095172	0.203637	-0.0775772	0.291443
LOGPRI	0.435372	0.318186	-0.832580	0.069745	0.105614
LOGFSDGDP	0.475170	-0.386652	0.210865	0.595284	0.475275
LOGDCBC	0.223007	0.841411	0.457620	0.177733	0.035915

Note: all variables in this matrix are in their logged form

Results show that among the above mentioned five principal components, 1st principal component describes 69.24% and 2nd principal component define 21.28% of the total variation. In contrast, the remaining three principal components (3rd, 4th and 5th) explain less than 10% of the total variation. This statistics of the variables express that the first principal component is the best measure of financial development in the case of Pakistan. The individual contribution of each of the variables of financial development to first principal component includes DCBC 22.30%, PRI 43.53%, M2 51.25%, FSD 47.52% and M3 52.17%, which will be used in the creation of general index used to measure the concept of financial development. In terms of descriptive statistics, results of this study show that all the variables; gross domestic product, foreign direct investment (inflows), electricity consumption and financial development (PCA) are usually distributed in accordance with findings of Shahbaz (2011) and Tang and Tan (2014). The corresponding Jarque-Bera statistics in our case also show the rejection of the null hypothesis of familiarity of the data. So we can conclude that the data is normal, and there is no problem of non-stability in it. However, the results of Descriptive statistics are mentioned in Table II as below:

Table 3. Descriptive Statistics

	LogEC	LogFDI	LogGDP	FD_Index
Mean	5.578277	-0.561833	10.53663	-1.22E-14
Median	5.783609	-0.491035	10.59052	-0.025332
Maximum	6.162068	1.299735	10.88559	3.155371
Minimum	4.660688	-2.786604	10.09989	-5.220132
Std. Dev.	0.484036	0.955344	0.235637	1.886341
Skewness	-0.595944	-0.173993	-0.279412	-0.421783
Kurtosis	2.001385	2.831337	2.119822	2.959841
Jarque-Bera	3.727487	0.230542	1.675790	1.099542
Sum	206.3962	-20.78780	389.8554	-4.55E-13
Sum Sq.Dev.	8.434484	32.85655	1.998900	128.0982

4.2. Results of Unit Root Test

The stability of data is further analyzed through implementation of augmented Dickey-Fuller (ADF). This test has been extensively used in existing literature like; Tang and Tan, (2014) and Shahbaz, (2012) for identification of trends in data of electricity consumption, financial development and economic growth. Results show that all the variables in their log form are stationary at first order and became stationary at 1% significance level. Table 3 displays the results of Unit root test of the study.

Table 4. Unit Root results

	Null Hypothesis: variable has a Unit Root Exogenous: Constant Lag Length: 0 (Automatic - based on AIC, LR, FPE, max lag= 2)			
	t-Statistics (Test Critical Values)			
Variable Name	**ADF value**	**1% level**	**5% level**	**10% level**
LogEC	-4.617903***	-3.632900	-2.948404	-2.612874
LogGDP	-4.394965***	-3.632900	-2.948404	-2.612874
LogFDI	-6.075929***	-3.632900	-2.948404	-2.612874
PCA_Index	-4.674176***	-3.632900	-2.948404	-2.612874

Note: All the variables are in Log form, instead of PCA_index because of having negative values in the index.
(***, **,* represents 1%, 5% and 10% significance Level.)

4.3. Results of Cointegration

The results of ADF test about integration shows that variables are stationary at *I* (1). This level of stationarity provides us a base for further examination of long-term relationship between the variables under study. For such purpose, this study applies the Johanson-Juselius cointegration approach among others, because of the fulfillment of the primary requirement about same level of integration of variables. In this cointegration test, the number of appropriate lags is selected on the basis of AIC, FPE, and LR test. Results show that the hypothesis of no cointegrating relationship is rejected at 5% level of significance, which means that there is a presence of long-term relationship between variables. With this view to find a long term relationship, this research further finds the failure of rejection of at most one cointegrating vector on the basis of both max Eigenvalues and Trace statistics. Hence we conclude that there is a significant long-term association ship between all the variables in the model, but the direction of such long-term causality is not acknowledged at this point. The results of this approach are mentioned in Table IV as following:

Table 5. Results of Cointegration

Hypothesis	Trace Statistics	Max Eigen Values
R = 0	51.42927 [0.0222]	33.73220 [0.0071]
R ≤ 1	17.69707 * [0.5885]	9.233686 * [0.8131]
R ≤ 2	8.463383 [0.4173]	8.404322 [0.3391]
R ≤ 3	0.059061 [0.8080]	0.059061 [0.8080]

Note: * represents the decision about number of cointegration relationship.
[ ] denotes probability distribution.

Statistics of the above-mentioned table shows that all the variables have some long-term interaction with each other. These statistics allow us to proceed towards the estimation of cointegration equation, which we found from the normalized cointegration coefficient matrix are as following:

$$LogEC = -0.82logFDI + 0.6358logGDP - 0.1639FD$$

Results of the cointegration are according to our expectation about the long run relationship. so, we can conclude that our results comprehend the possessions of both Shahbaz and Feridun (2012) and Alam (2013), who reports the occurrence of long-term cointegration relationship between economic growth and electricity consumption. Our results also support the findings of Sadorsky (2011), who state that the financial sector developments raise the demands of big tickets in the economy, which raise the demand for electricity.

4.4. Vector error correction model

The direction of causality can be determined through application of vector error correction model. ECT approach is adopted in this research because exclusion of ECT from model gives unrealistic results (Adjaye, 2000). Another reason for this application is the segregation of both long and short term causality in this model. Table V provides an overview of the VECM results.

Table 6.VECM Results

	Short term relationships (Wald Test Statistics)				**Long term relationship**
Variables	**LogEC**	**LogGDP**	**LogFDI**	**FD (PCA_index)**	**ECT_{t-1}**
LogEC	---------	0.430138 [0.8065]	2.944502 [0.2294]	1.802147 [0.4061]	0.084167 {0.02111}
LogGDP	6.541744** [0.0380]	----------	0.405112 [0.8116]	8.928322** [0.0115]	0.635782 {0.79426}
LogFDI	5.026477* [0.0810]	1.086044 [0.5810]	----------	1.370376 [0.4050]	-0.814808 {-3.45348}
FD (PCA_index)	9.965790*** [0.0069]	5.772555* [0.0558]	8.706936** [0.0129]	----------	-0.163883 {-2.83130}

Diagnostic Tests		
R^2	0.6031	
Adjusted R^2	0.4541	
F-Statistics	4.0506	[0.0029]
JB (χ^2 Normality)	0.8207	[0.6634]
BG (χ^2 Hetero)	0.1156	[0.9438]
BG (χ^2 Serial)	3.5211	[0.9906]

Note: (***,**,* represents the significance level of 1%,5%,10%)
[] shows the probability value, { } shows the value of standard error

Results demonstrates that the error correction term is positive and significant, which means that in short-term, there is a positive relationship between financial development, electricity consumption and economic development. These results are in accordance with the trends observed in Pakistan. For example, we see that whenever the politicians receive aid for Saudi-Arabian government in holy month of Ramadan, the government ensures the uninterrupted power supply for the month. As the reserves of aid finishes, the system again comes back to its previous stage of Power shortage. In contrast, the speed of adjustment is also negligible and rank up to 8.47%, which further comprehend the results of error correction term. This means that in short run the system will not overcome its main problem of power shortage.

So we can conclude that there is a need of long term planning to grab this evil, which shows that in the long-term causality moves from electricity consumption towards financial development, FDI inflows and per capita GDP. This direction is in accordance with the basic argument that if the country ensures the adequate availability of power resources to the industrial sector along with handsome environment, investor attracts towards this nation and prefers to invest their capital in the economy.

In contrast, the chi-square values obtained from Wald statistics demonstrate the presence of short-term causality running from financial development towards electricity consumption, FDI inflow towards electricity consumption and from GDP towards electricity consumption. 1% rise in electricity consumption can be observed if there will be a rise of 5% in FDI in the country. Similarly, if GDP per capita raise by 6.54% then the electricity consumption will increase by 1%. Furthermore, the rise of 9.5% in the financial sector development would cause a 1% rise in the value of electricity consumption. On the other hand, both financial development and GDP cause each other at the same time, which shows the existence of bidirectional causality running between GDP and FD.

All of these results give strength to the argument that Pakistan is an energy dependent country and needs electricity for its long term growth. So, any energy policy which is aimed to conserve the energy will have not much significance on the economy, become conservation of system where raise the power supply to industrial sector, also reduce the supply to domestic sector. This situation leads towards unrest in the economy, which directly hurt the business in the economy. When no electricity is provided to the domestic users their demand of big tickets is entirely changed, also badly hurt the business in the domestic industry. This reduction where reduce the turnover of the investors in the market, also reduce the per capita income by raising unemployment in the economy.

On the other hand, our results regarding bidirectional causality are in accordance with Iqbal, Shaikh and Shar (2010), which states that this causality is the evidence of different growth theories. These theories state that the attraction of FDI in this economy gives fruitful results for the economy. In contrast, previous trend FDI further supports the hypothesis that FDI inclusion raises the level of development in the economy.

5. Conclusion and Policy Implications

The intent of this study is to find the role of financial development in traditional electricity-growth nexus. Johanson Cointegration, Principal component analysis and vector error correction methods were adopted in this study to define the existence as well as direction of both long and short term causality between the variable of this study. Analysis of the study determines the existence of long-term causality from electricity consumption towards financial development and economic growth. On the other hand, short term causality is observed from FDI, FD and GDP towards electricity consumption in the economy. Further a bidirectional relationship is also observed between FDI and GDP, which supports the findings of Iqbal et al. (2010).

Pakistan is well known for its agricultural land and its viable geographic as well as market situations. These markets have enough potential to provide sufficient revenue to investors. A number of different factors have been observed in previous decade that reduce the potential of these markets. One of such limiting factors is the power shortage in the economy, which retards the uplift of the economy. This shortage has been further observed as a reason of capital flight from Pakistan towards other nations, which results in raising unemployment as well as reducing interest of foreign investor in the economy. In order to overcome this problem, different governments take various short term steps in their tenure with a view to raise their popularity. Unfortunately, all of these efforts became vague and imposed further debt burden instead of dwindling of this problem. So in present, where Pakistani faces lots of challenges, one main challenge for the government is to overcome the power shortage problem, so that the economy regains its old position in the race of developing nations. In this crucial movement, there is a need of proper efficient long term planning, which helps in the eradication of this problem. The Government of Pakistan have to leave the traditional point scoring strategy and collect all the school of thoughts at any single point of reference, so the appropriate enactment will be conceivable.

On the basis of the results of the study, I hereby suggest that the government should decentralize the power sector and initiate the credit expansion program in the power sector. Apart from the expansion there is a need of provision of incentives to attract the level of investment in the economy. The attempts of the current government are impressive, but the involvement of credit expansion program as well foreign investment raise the numbers of competitor in the power sector. The result of such competition will be technological advancements with a number of substitute ways, which will help the economy to nib this devil of load shedding in the bud. Furthermore, governments instead of the provision of short term yellow cap scheme and youth employment loan schemes initiate technical education programs, which will provide us an educated and well trained class of labor. So that the investor moves

towards economy and enjoys the benefits of the abundant supply of power as well as huge profits, which in turn raise the movement of the nation towards economic uplift.

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A Conjecture Concerning Prime Numbers

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Abstract. The paper examines the conditions under which a second degree polynomial generates primes variable values between 0 and the constant term - 1. It is shown that for values of the constant term equal to or less than 41, such polynomials exist and we are proposing a conjecture that for the polynomials with the constant term greather than 41 the statement is not true.

Keywords: prime; polynomials

1. Introduction

The prime number theory dates back to ancient times (see the Rhind papyrus or Euclid's Elements).

A number $p \in \mathbf{N}$, $p \geq 2$ is called prime if its only positive divisors are 1 and p. The remarkable property of primes is that any nonzero natural number other than 1 can be written as a unique product (up to a permutation of factors) of prime numbers to various powers.

If there is not a formula, for the moment, generating prime numbers, there exist a lot of attempts (all unsuccessful) to determine it.

Unfortunately, many results about primes are at the stage of conjectures (theorems that seem to be valid, but remained unproven yet).

2. Main Results

Let $P(n) = an^2 + bn + c$, $a,b,c \in \mathbf{Z}$, $c \geq 2$, $(a,b,c) = 1$.

We propose the determination of $a,b,c \in \mathbf{Z}$ such that $P(n)$=prime $\forall n = \overline{0,c-1}$.

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Using the Wolfram Mathematica software, in order to determine the polynomials (for c≤41):

```
Clear["Global`*"];
polynomial[a_,b_,c_,n_]=a*n^2+b*n+c;
lowerlimitc=2;
upperlimitc=41;
upperlimit=200;
For [c=lowerlimitc,c<=upperlimitc,c++,
        maximuma=0;
        maximumb=0;
        maximumc=0;
        maximumcounter=1;
        increase=2*upperlimit;
        If[PrimeQ[c],
                For[a=-upperlimit,a<=upperlimit,a++,
                For[b=-upperlimit,b<=upperlimit,b++,
                        counter=0;exit=0;
                        If[a≠0,For[n=0,n<c,n++,
                        If[PrimeQ[polynomial[a,b,c,n]]&&exit=0,counter=cou
                nter+1,exit=1]]];
                        If[exit=0&&counter≥
                maximumcounter,If[(Abs[a]+Abs[b])<
                increase,maximumcounter=counter;maximuma=a;maximumb
                =b;
                maximumc=c;increase=Abs[a]+Abs[b]]]]]];
If[maximumcounter≥2, Print["a=",maximuma," b=",maximumb,"
c=",maximumc];
For[n=0,n<c,n++,Print[polynomial[maximuma,maximumb,maximumc,x]]],If[
PrimeQ[c],Print["c=",c," accepts no polynomial"]]]]
```

we get:

c	$P(n) = an^2 + bn + c$	Module of Primes
2	$P(n) = n^2 + 2$	2,3
3	$P(n) = n^2 - n + 3$	3, 3, 5

5	$P(n)=n^2-n+5$	5, 5, 7, 11, 17
7	$P(n)=2n^2-2n+7$	7, 7, 11, 19, 31, 47, 67
11	$P(n)=n^2-n+11$	11, 11, 13, 17, 23, 31, 41, 53, 67, 83, 101
13	$P(n)=6n^2+13$	13, 19, 37, 67, 109, 163, 229, 307, 397, 499, 613, 739, 877
17	$P(n)=n^2-n+17$	17, 17, 19, 23, 29, 37, 47, 59, 73, 89, 107, 127, 149, 173, 199, 227, 257
19	$P(n)=2n^2-2n+19$	19, 19, 23, 31, 43, 59, 79, 103, 131, 163, 199, 239, 283, 331, 383, 439, 499, 563, 631
23	$P(n)=3n^2-3n+23$	23, 23, 29, 41, 59, 83, 113, 149, 191, 239, 293, 353, 419, 491, 569, 653, 743, 839, 941, 1049, 1163, 1283, 1409
29	$P(n)=2n^2+29$	29, 31, 37, 47, 61, 79, 101, 127, 157, 191, 229, 271, 317, 367, 421, 479, 541, 607, 677, 751, 829, 911, 997, 1087, 1181, 1279, 1381, 1487, 1597
31	$P(n)=-3n^2+45n+31$	31, 73, 109, 139, 163, 181, 193, 199, 199, 193, 181, 163, 139, 109, 73, 31, 17, 71, 131, 197, 269, 347, 431, 521, 617, 719, 827, 941, 1061, 1187, 1319
37	$P(n)=-4n^2+76n+37$	37, 109, 173, 229, 277, 317, 349, 373, 389, 397, 397, 389, 373, 349, 317, 277, 229, 173, 109, 37, 43, 131, 227, 331, 443, 563, 691, 827, 971, 1123, 1283, 1451, 1627, 1811, 2003, 2203, 2411
41	$P(n)=n^2-n+41$	41, 41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601

For $c\geq 43$ we have that from the condition that $P(n)=an^2+bn+c=\text{prime}\ \forall n=\overline{0,c-1}$ implies that a+b=even. Indeed, if the prime is different from 2, we have that: $an^2+bn=\text{even}$ for any $n=\overline{0,c-1}$ except possibly two values for which $P(n)=2$.

But for two consecutive numbers n, n+1 such that $P(n)=an^2+bn+c=\text{prime}_1\neq 2$, $P(n+1)=a(n+1)^2+b(n+1)+c=\text{prime}_2\neq 2$ we have that: $an^2+bn=\text{even}$, $a(n+1)^2+b(n+1)=\text{even}$ therefore the difference: $2an+a+b=\text{even}$ that is $a+b=\text{even}$.

The structure of the polynomial becomes then:

$$P(n)=an^2+(2d-a)n+c\ ,\ a,b,d\in\mathbf{Z}$$

Investigating for values of c greater than 41 to 1000, we have not found polynomials with this property, considering reasonable limits for a and b less than 200.

Finally we state the following:

Conjecture

There is not a polynomial $P(n) = an^2 + bn + c$, a,b,c$\in$**Z**, c$\geq$43, $(a,b,c) = 1$ such that $P(n)$=prime $\forall n = \overline{0, c-1}$.

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New Institutional Economics and Economic Development of the Republic of Kosovo

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Abstract: The purpose of this study is the analysis of the new institutional economics according to parameters and criteria that affect the growth and economic development such as free movement of capital, contract enforcement, information costs, risk transfer costs, free competition and their application into practice. Institutional innovation, in terms of establishing more efficient and effective institutions, can only be done if there is support of the whole society, but given the limitations set forth in Kosovo, as possible change of the Constitution, some laws and other internal and external restrictions, the achievement of this goal is difficult. Because of the very specific and serious past, and also very long delay in the process of transformation and transition, the economic development and new institutional economics in Kosovo, according to almost all development indicators has significantly stagnated in comparison with the countries of the European Union and Western Balkans. Economic growth and development implies a very complex and multidimensional process, influencing many factors such as economic, technological, institutional, political, social and cultural. Economic growth depends on political institutions and their capacity to define in a more clear and acceptable way its common goals. Given the political dimension of the capacity of the state, certainly should be considered also the institutional approach of development i.e. political dimensions of the institutions that support economic development.

Keywords: new institutional economics; economic thought; economic development; transaction costs; property rights

JEL Classification: D23. E02, F43, O11

1. Introduction

Contemporary theory of economic development has broaden our knowledge by explaining of why some countries are rich and other countries are poor, by analyzing institutions, namely the norms of behavior that enable the acceleration of economic growth. Within the new institutional economics, the institutional aspect of economic growth increasingly is being treated, and particularly the origin of some institutions. What are those institutions that accelerate economic growth, and which are the ones that slow the economic growth? Why in some countries have been created certain

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institutions that bring higher efficiency, and in some other countries institutions have much lower efficiency?

These questions and dilemmas, respectively the research to find answers to these questions, in recent years are bringing more and more together the economics science with the justice science.

A well-regulated legal system in a country represents a value for both, economists and lawyers, but their motivation up somewhere is different. Lawyers, view the system as a value of its own, while for economists such a system has institutional value, since such regulation ensures maximizing economic efficiency and social Welfare. At first we must emphasize that the current institutional environment is not suitable to sufficiently affect the proper and adequate acceleration of growth and economic development. For states that are under transition period, representatives of this field present and provide adequate proposals that will affect the acceleration and strengthening of institutional innovation, which in the short term will provide an efficient legal system, an efficient competition law, consumer protection and an effective regulatory system that will affect the creation of a safe and sustainable financial system.

Key terms on which the new institutional economics relies, are: institutions, economic development, capital, contracts, etc.. Within this concept are presented different dilemmas: why some countries in a successfully and highly efficient way are organizing their economies, while other countries are failing to realize the growth and economic development, or achieving an insufficient economic development that does not provide demographic investments? Numerous authors are arguing that the answer to this question is primarily determined by the importance and the role of the institutions in the economies of these countries.

2. New Institutional Economics and Economic Development of Kosovo

Because of the very specific and serious past, and also very long delay in the process of transformation and transition, the economic development and new institutional economics in Kosovo, according to almost all development indicators has significantly stagnated in comparison with the countries of the European Union and Western Balkans (*Based on public institutions index, which is estimated by The Global Competitiveness Report. Kosovo is not on the list of countries analyzed, but the Western Balkan countries have a lower ranking).

After World War II onwards, within the theory of economic development are profiled mainly two economic concepts, the first, the traditional concept of economic development that emphasizes mainly the primary importance of the state as the initial instigator of the development processes and secondly, neoclassical concept that emphasizes the crucial role of the market, prices and different stimulants of

economic development.

After the destruction of the socialist-centralist system, almost on all former socialist countries has dominated the neoclassical concept, where all states of the region, as well as Kosovo after 1999, were treated by centralist-planning virus through transitional standard package. This transitional package was based on the principles of the Washington Consensus, which is prepared on the basis of the neoliberal model of economic development and the same was strongly supported by the IMF and World Bank. This model in most transition countries was based on the full operation of the free market, to which is achieved through rapid and massive privatization. Recent years many experts and representatives of new institutional economics emphasize that the models that offers the neoclassical school of economic development, noting hypothetically that production, economic growth and the differences between industrial countries and developing countries can be explained by focusing on basic elements (resources), technology and preferences are not sufficient for good and efficient economic results.

Neoclassical theory in most cases was an inappropriate tool for analyses and for determining economic policies that will promote efficient economic development. Relying on efficient markets to foster efficient allocation of the vast majority of resources, the neoclassical economics does not reflects many institutional requirements of particular importance to enable the creation of such markets, calculating that rational decision makers in certain markets due to the action of the information, will improve their previous decision, if the same is proved to be wrong. This is further strengthened also by the conclusion of one of the most prominent scholars of the modern economy, Stiglitz, who points out that "Self-regulating market is not able to respond to these requests. This was confirmed by the contemporary economy, in which in the period of dominance of the free market - market fundamentalism, the state has supposedly saved many times the self-regulating apparatus of the financial system, eventually the financial collapse in 2008" (Beroš, 2013).

Long dominance of neoliberal concept, in most cases regarded as a withdrawal of the state from the economy, as well as reducing the political and institutional capacities in order to implement the control of economic processes. Typical neoliberal formulas in most transition countries are implemented through the full liberalization of financial markets, liberalization of the labor market, complete removal of restrictions on flows of goods and capital, increasing the participation of foreign capital, privatization of the public sector, and a restrictive financial and monetary policy.

But, on the other hand, is emphasized that efficient markets are a function of institutions that enables measuring and enforcing contracts with as less costs, ie. as how the transaction costs are lower, so and markets will become more efficient (Shavell, 2009, p. 291-330).

Understanding the institutions can be defined that institutions are the system of established rules and the social dominant rules that structure the social interactions. In most cases, institutions as a term is used in the sense of defining the organization as Ministry, state agencies, administrative bodies, universities etc., but in this treatment will be used in a slightly different meaning. The impact of institutions on economic development can be determined after determining the two meanings, (1) that institutions are important for economic development and (2) that the analysis are possible through the economic theory instruments.

3. The Impact of Institutions on Economic Development

Institutions are essential for economic development, which help explain market failures and provide explanations for the role of the state in the economy. Institutions consists of formal constraints, such as constitutions, laws, rules, property rights etc. and informal constraints, such as the code of conduct, agreements, self-imposed rules of conduct etc. (North, 1994, pp. 359-368).

Measurement of institutions includes: measuring the quality of institutions, social capital, social and political features and political instability. Numerous researches made in this field have verified that the quality of institutions and social capital have positive correlation with economic development, while social features and political instability have negative correlation, while for the political features of institutions is not defined the significant interdependence in relation to economic growth (Aron, 2000, pp. 99-135).

If institutions are defined in such a way that will stimulate the economic activity and increase productivity, then this institutional arrangement, indirectly by allocating the resources, will stabilize the economic growth. Productivity as a ratio between inputs and outputs of production will increase if achieved to reduce overall costs, transactional costs and transformative costs. The first, relates to the measurement and enforcement of contracts and the second is related to the physical costs of production. Transactional costs are reduced through institutional changes, while transformative costs are reduced through innovative technological change. Institutional and innovative changes affecting reduction of transactional costs (and with it also the increase of productivity, and in the last instance in the economic growth) are those institutions that enhance the movement of capital, reduce information costs, reduce the costs of risk transfer, and increase the efficiency of contract enforcement.

Building genuine institutions that enable and stimulate the free movement of capital, gains even more importance due to the law of the tendency of the rate of yields to fall and simple account that in a country where there is more capital in proportion to the number of inhabitants, capital yields will be decreased. Since the lack of capital in developing countries affects and is reflected in increased yields of capital, then

increased of its movement is crucial (Stiglitz & Driffill, 2000, p. 741). The development of the capital market and the ability of the market to more efficient allocation is a key precondition for meeting the previous criteria. Although the free movement of capital and investments (including individuals, goods and services) is guaranteed by the Constitution of the Republic of Kosovo (article 119.2 dhe119.6), this is not happening during these years, primarily due small market that Kosovo has, no confidence of the population, the lack of public information, the non-existence of adequate legislation etc. (capital market in the Republic of Kosovo was opened for the first time in 2012 with Regulation of Ministry of Finance-CBK no. 01/2014 for the primary and secondary market securities of the Government of the Republic of Kosovo and over the years many auctions were held with the participation of primary stakeholders, but turnover is symbolic and with very short maturity).

4. Institutional Environment, Risk Transfer and Contract Enforcement

The institutional environment appears also as a constraint to successful management. Informal institutions, in most cases in different analyzes are taken as input and provides the institutional space in which they operate, such as customs, habits, traditions, norms, cultures etc. But, in this case also the religion is of particular importance. This level of informal institutions is often outside of the economic and political scope.

The institutional environment that occurs through formal institutions is the first level of economization that should be changed as soon as possible, while the management, dealing with the last unit of economic action and it is the transaction. At this level of reality, in the practical way is reflected functioning of the social and economic system. Often times, since also the actions of formal institutions are not perfect and free of charge, participants in the transaction are oriented in a private and alternative solution aiming to resolve the dispute and that in the form realize mutual benefit (notary services, private bailiffs, arbitration, bankruptcy administrators, etc.).

In most cases, changes in institutional environment are developed more slowly than changes at the management level. An institution to change in general, it is not enough to change only the formal rules, which is in the state authorities responsibility, but should also influenced in informal rules, which are created according to the specific logic and dynamics, but on the other hand, should not left aside the application of these rules.

Demands for the efficient economic institutions cannot be created timely, which largely confirms the continuous demand for this type of institutions in the Western Balkans countries and their very fragile success in creating these institutions. Many of these countries for years are waiting to put the adequate policies and transform existing institutions from which will benefit their citizens.

The main sources of institutional change according to Douglas C. North, are relative price changes and change of preferences (North, 1997).

In the period so far in Kosovo, the international community on numerous occasions, by wanting to change the parallel inherited institutions or those created in the last decade of the 20th century, they often have used the mechanism of relative price change (through the blocking or permitting of economic aid or donations, or through setting conditions for membership in regional or global organizations), or some decisions that were in full competence of local institutions, but that cannot be implemented by them. In the event that local decision makers would not agree fully with the international community requirements, according to this logic, this approach would have a limited success. Changing the formal rules (laws) is the ultimate goal of approach of changing the institutions. But on the other hand, to make this policy effective, special attention should be paid to informal constraints and enforcement mechanisms. Since 1999, Kosovo was administered by UNMIK, while the Self-Government bodies and the Parliament had limited competences, while The Fourth Pillar of UNMIK was the main decision-maker and responsible for increased economic development. In 2002, it founded the Kosovo Privatization Agency (KPA), where its primary duty was the privatization of socially owned enterprises, in which process and decision-making on the KPA Board, the dominant role have had the international representatives.

From the perspective of economic policy, it can be said that the process of institutional change is greatly under the influence of force structure between different groups of interest, then the cultural and religious tradition. These factors depending on the proper orientation of actions can influence that certain institutional rules more easily be accepted. As stable institutions can be considered only those institutions that have the support of the whole society, which often can hardly be achieved, especially in the Western Balkan countries, in particular in the Republic of Kosovo, in which the ethnic divisions still remain very pronounced.

Risk transfer costs in different transactions can be very high, if it is carried out only by one subject or contracting party. For this reason, the importance of developing the capital market consist on the intention of diversifying the risk in smaller portions and extending to as many entities or participants in a certain market. Risk of fire and costs of insurance from possible fire, e.g. to a person (landlord) will be very high, if he as a person will cover the whole eventual costs. But, with the purchase of an insurance policy against a possible fire, he will share the possible risks and costs with another market participant, which in this case is the insurance company. Presentation and distribution of risk in the capital market usually takes place through the Joint Stock Companies.

Developmental projects, the construction of manufacturing facilities and projects of new lines within the existing production capacities are expensive and quite risky, and on the other side it has no sufficient guarantees that through market participation will be achieved safe return from the invested funds. This is the main reason, and

also most individuals do not have sufficient funds but also the courage to face with such a project. But when a big investment, is distributed in larger numbers of individuals (Shareholders), then the overall risk will be shared between the shareholders, but also the incurred costs will be easily covered.

Cases dealing with risks, the economic theory puts them in direct relation to the development of the capital market, in which case the rapid development of capital markets would enable the highest level of risk management, and the opposite. Faster and efficient implementation of contracts, resolving business disputes, etc. does significantly influence the investment decision-making. Judicial system of the Republic of Kosovo is not currently providing a quick and efficient resolution of business disputes, for various reasons, such as inefficient processing of cases, primarily due to the accumulation of many cases from the past decade that have not yet been resolved, inadequate management of courts and prosecutors' offices at different levels, lack of professional staff with adequate experience in economic and financial fields etc.

Freedom is not only the basis of assessment of success and failure, but is also a key determinant of individual initiative and social effectiveness. The alternative of strong enforcement of the law is the existence of mutual trust and mutual separation of responsibility and commitment (Sen, 1999, p.39).

Preservation and enforcement of contracts often have been exacerbated or prevented in situations where the morality of the market and the business confidence are dysfunctional and nonexistent. Viewed in the long run, the state should engage in more extensive promotion of these two “business values” that are part the informal institutions. So far in Kosovo is still not provided any measure of social values that will be supported by the existing political system.

5. Concluding Remarks

Institutional innovation, in terms of establishing more efficient and effective institutions, can only be done if there is support of the whole society, but given the limitations set forth in Kosovo, as possible change of the Constitution, some laws and other internal and external restrictions, the achievement of this goal in Kosovo is very difficult. In what form the support should be created for the establishment of these efficient, consistent and stable institutions?

New institutional economics elaborates that the way to achieve this goal has to do with action in informal institutions, affecting in various measures to increase the trust in institutions, and this goal can be achieved in two ways: first, the continuous explanation of the mutual interdependence of personal and common benefits, and their indivisibility to all citizens of Kosovo, and secondly, on increasing the costs for

the negative impacts of certain individuals or groups of interest in the stability of institutions.

In relation to the formal institutions, the responsibility of every Government should be building institutional infrastructure, i.e. providing of efficient legal system, effective laws in all segments of society, efficient regulatory system that would ensure a safe and sustainable financial system.

Some key indicators of the state of the Republic of Kosovo institutions have a tendency of a slow improvement, but at a lower level compared with other Western Balkan countries, such as protection of property rights, legal certainty and guarantee of signed contracts, protection of intellectual property rights, guaranteeing free competition, consumer protection, burden of state regulatory, etc.

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The Strategic Choices of Small Medium-Sized Enterprises Integration: Evidence from Specific Economic Territory

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Abstract: The emergence of SMEs was conditioned by institutional and economic factors. These factors, as well as other internal promoted the integration of strategic thinking within this type of business, by adopting suitable strategies. Indeed, our research objective is to determine the extent of adoption of the approach and the strategic decision by the Algerian SMEs, and highlight the characteristics of its strategies through a survey on a sample of SMEs located in the West of the Algeria. The analysis of this empirical study will be preceded by a theoretical overview that aims to show the different concepts of business strategy, and more specifically those of the SMEs, and then to identify the specifics of strategic management within the SME. Finally, we will analyze the survey data basing on graphics to determine the different strategic options available to theses SMEs from the sample according to this specific economic territory.

Keywords: SMEs; integration; strategic approach; survey; business strategy; strategic options

JEL Classification: L21; L53; P25; M13

1. Introduction

In recent years the emergence of SMEs to global was favored by the failure of most large companies, on the one hand, and on the other hand, by changing relationships between big business and small business. For the Algeria this type of business is a large part of the economic fabric, where its importance which is increases more and more because of its role in the socio-economic development.

Its emergence was conditioned by institutional and economic factors that were synchronized by the economic opening of our country on the international environment. These changes have intensified competition by putting this type of business in vulnerable situations. Thus, Algerian SMEs underwent changes in its mode of operation, especially with the advent of new entrepreneurs that use all the

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modern techniques of management.

However, these internal and external factors have promoted the integration of strategic thinking within the Algerian SMEs. By adopting suitable strategies, SMEs can achieve goals in a changing environment and evolving, so these strategies will foster the SMEs to have competitive advantages. Of the foregoing, we can ask the following question: "what kinds of strategies are adopted by SMEs. And what is the reality of this approach for SMEs in Western Algeria? ». This paper is structured as follows:

To answer our question, it conducted an empirical study on a sample of Small medium-sized Enterprises (SMEs) located in the West of the Algeria to show the characteristics of strategic management in this type of business, developing a typology of SME according to strategic management adopted. This study is preceded by a presentation of the theoretical approaches of the strategy and the SMEs.

2. Literature Review

2.1. Definition of the Firm's Strategy

It is difficult to give a unique and universal corporate strategy definition, as each author has a definition that own according to his approach to research. For Igor (Ansoff, 1989, p. 52), "the strategy is one of decision criteria that guide the behavior of an organization" (Ansoff, 1989), and according to wear (Porter, 1989, p. 21) the company's strategy is: "the combination of the purposes (objects) that strives to reach Firms and of the (measures) means by which it seeks to achieve. (Porter, 1982)

2.2. The SMEs

For the definition of SMEs, there are two approaches, definitions using qualitative criteria and those using quantitative criteria. According to the General confederation of small and medium-sized enterprises, the SMEs is "a unit of production or distribution, a unit of direction and management, under the authority of an entirely accountable Manager for the company, which is often owner and which is directly related to the life of the Enterprise"[1] (GCSME, 1983, p. 1).

Through this definition, we can say that the qualitative definition is based on the characteristics of the SME and its leader, the relationship between the latter and the

[1] General Confederation of Small and Medium-sized Firms, service of documentation, 1983.

Enterprise. Concerning our research we are based on the definition adopted in Algeria considers that:

- The SME is an enterprise of production of goods and/or services;
- From (1) to (249) employees;
- Whose annual turnover does not exceed 2 billion dinars or who's annual. balance sheet total does not exceed 500 million dinars.

2.3. The Main SMEs Strategy

According to other authors (i.e. Horovitz, & Pitol-Belin, 1997, p. 187) give a definition of the SME strategy which is a "Basic Concept of general policy, the strategy has two goals: (I) choose among the alternatives, the guidelines that the company wants to take for its future; (ii) ensure the consistency of agreed actions[1]. Indeed, according to (Chaillot, 1995, pp. 57-58) the notion of the SME strategy translates into: "the resumption of ideas - aims - strongly implanted.

Where the main idea is to be the market leader, this translates concretely by growth of turnover objectives, increase its market share, and the second idea is the need to innovate to win, thus, the strategy developed by the leader in this type of enterprise, is limited to a process of decision often not formalized and not communicated", but rarely used as a management tool. So, the strategy of SMEs does not differ from that of big business, but it has characteristics, such as non-formalization in its formulation.

2.4. The Pillars of the Strategy for Small Business

According to (Marchesnay & Fourcade, 1997, p. 80), the strategy of the SMEs can be grasped by four-dimensional (Figure 1) which are:

[1] Quoted by: Marchesnay, Strategic Management, the publishing (Edi) of organizations, 1997.

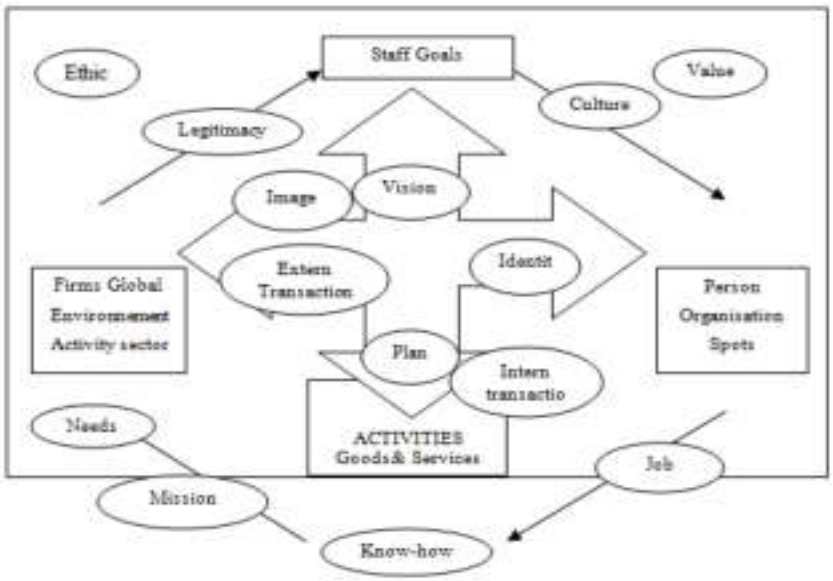


Figure1. The SMEs strategic system

(Source: Marchesnay & Fourcade, Management of SMEs/SMIs, Nathan, 1997, p. 80)

The objectives According (Thietart, & Xuereb, 1984, p. 38) to the lens give a direction to follow and influence the type of actions to be undertaken, it constitutes a step of its evolution, and then it is used to assess and monitor the company. For a goal to be useful for the Organization, it must be clear, specific, realistic, consistent understandable, acceptable and controllable. (Thietart, & Xuereb, 2005) The setting of objectives is shared between two schools, advocates a normative search methods to determine targets, one more descriptive, merely to observe the various practices (Helfer, Kalika & Orsoni, 2006, p. 57) which vary according to the circumstances. Concerning SMEs, there is a strong articulation between its objectives and those of its leader.

However understanding of management style (Gervais, 2003, p. 40) and goals of the SMEs, are carried out through the knowledge of the profile of the entrepreneur. Looking for independence managerial and financial, is one of the main concerns of the contractor, and next to these objectives, others can be searched for example: the desire to ensure an income, or a job for his family, the desire to produce products of quality, creativity, power, status, self-realization, the sustainability of the enterprise, the continuity of family traditions.

The objectives of each contractor are often multiple (Janssen, 2009, p. 95) and are likely to vary with the wire of time and the development of Enterprise. Julien & Marchesnay (1992, pp. 58-59) offer two types of entrepreneur according to their

objectives, where this typology is based on three major socio-economic aspirations, with the contractor, which are: the sustainability of the company, involving in particular, hoping to transmit it to a member of the family, independence and growth.

2.4.1. The Organization

On the one hand, the organization is a community of people together to achieve the objectives set, and one on the other hand the Organization refers to action to organize which consists to assign tasks to each worker, distribute responsibilities, delegating authority, and develop a communication system. However, these two approaches are complementary, for example: the increase in the size of the organization needs to recruit more workers, implying a reorganization of tasks, authority, and communication system (Marchesnay & Fourcade, 1997, p. 89). Indeed, the Organization as a structure, characterized by several parameters which are specific in SMEs, citing (Marchesnay & Fourcade, 1997, pp.190-191) among other things:

- A low specialization of tasks: resulting from the lack of formalization and the definition clarifies workstations that, often, workers have a variety of different tasks, and have no well-defined position;
- A strong centralization: because the powers are in the hands of the leader of SMEs, where the latter is a key player in the Organization, but it does not some leaders to delegate a part of their authority to officials, and even some other, develop a team spirit in their business, all this depends on the profile and skills of the members of the company;
- A low degree of formalization: the structure is informal, because usually the leader of the SME assigns no rules, procedures, and methods of written communications, but it may expand them in the case of recruitment of an officer, or when an SME is embarking on an operation of certification;
- A flat structure: which results from the reduction of hierarchical levels, as a result, a large number of employees are placed under the control of a same supervisor, and when the range of control becomes very large, the leader of the company delegates some of its functions to a Deputy. But this type of structure is not the case of all SMEs, where according to (Marchesnay, 1998, p.73), the organizational configurations for SMEs are very diverse, and range from crafts to the most sophisticated structures.

1.4.2. The environment Factor

It can be decomposed (Porter, 1989, p. 4) into two levels: the first is societal which includes communities and local institutions of all types, as well as the network of personal relationships woven by the leader, and the second is industry which can be divided into several levels:

- The industry which includes the activities in which operate the business, this level according to wear is limited by the following actors: direct competitors, indirect (incoming potentials and substitutes) customers and suppliers (Porter, 1989, p. 4) so the institutions that have a role in the regulation and the regulation of the industry. For SMEs, this level includes an assessment of vulnerability.
- The sector includes all of the actors who provide the chain of processing and adding values to a product, or a set of related or interconnected products, since the initial stages until the final stage, with dependency in terms of concentration, substitutability and essentiality analysis. This dependence is analyzed as the degree of reduction of strategic freedom due to a relationship with a partner: highly concentrated in the stream, hardly substitutable and essential for the activity of the Enterprise.
- The market, on which is exerted direct competition, which may prove more to the less aggressive.

2.4.3. Activity Stage

The activity basis corresponds to goods and services offered by the company to its customers. (Marchesnay, 1991, p. 53), can see that small business can develop activities where it has a competitive advantage over big business. Contrast of big business who is looking for a size higher, in relying on three elements which are: the economy of learning and experience, economy of scale and scope economy, small business is a breeding ground, so far as it remains small in the following activities: economies of experience (innovative and creative activities), economies of scale (high load in services activities), savings of field (strongly segmented niche).

2.5. The SME strategic approach

For (Helfer, Kalika, & Orsoni, pp. 429-430), “whatever the status of the small business, it adopts the strategic management, and its head has a preliminary view on the future of the Organization in the relatively distant future, but this view is not written, formalized, as it is in big business, so the managerial small business diagnosis has singularities”.

Thus, (Saporta, 1998, p. 108), who sees that the choice of a strategic option, goes through a planning process, and after that it examined the attitudes of the leaders, said that these behaviors are not combined with the approach, where 'the drafting of a written strategic plan always seems to be the exception rather than the rule among leaders of SMEs'.

In the same vein (Thurston, 2001, p. 99), stipulates that: “planning, for some SMEs, is not explicit, where leader simply has it in his head, for others, it is formal and

takes the form of texts carefully prepared objectives, specific actions, and corresponding budgets". Indeed, the same author said that the adoption of a method or the other depends on the following factors:

- Style and abilities the boss;
- The degree of involvement of other people in management decisions;
- The complexity and uncertainty of the activity, indeed, according GERVAIS the union of these factors favor (Gervais, 2003, pp.414-416) the establishment of planning in an SME as a system of formal planning, can not be necessarily a guarantee of success for this type of business.

Despite that there are difficulties in the implementation of the planner approach in SMEs, (Saporta, 1997, pp. 24-31) has some suggestions for the strategic approach will be adapted, which revolve around: minimizing resource consumption, including available time officer; consistency with the 'strategic style "intuitive, sometimes dazzling, officers involved, particularly in the context of the opportunities as they come in; assistance given to the leader to have a long-term business and design options to steer it in the context of this vision.

Indeed, the strategic process in SMEs is not always planned, and is described (Mintzberg, 1998, p. 24) as follows, "the policy process in an entrepreneurial[1] organization, often visionary type largely deliberate, but emerging and flexible details, as is intuitive and often facing an aggressive looking for opportunities". So, we can say that the strategic approach of the SME is related to the character of its leader.

2.6. The SME Strategic Options

In terms of strategic options (Saporta, 1997, p.6) ordered sequentially the various options available to SMEs, which are designed to maximize the chances of survival.

2.6.1. Innovation Strategy

The purpose of business survival, (Thietart & Xuereb, 2005, p. 124), stipulate that a strategy of innovation is probably the option for organic growth, the external effects are most important include: the structure of competition - the emergence of new competitors - the growth and creation of new business segments.

The small Enterprise often innovates through a pragmatic and non-systemic application. In other words, SMEs will innovate when driven by its dynamism, facing opportunities or new market needs, with constraints to reduce manufacturing

[1] As we have said before, most SMEs have an entrepreneurial structure.

costs, or face heightened competition. Also, the resulting new technology innovation is a productivity improvement factor and they can maintain or increase competitiveness especially in little or unprotected industrial branches.

The innovations of SME experiments show that their efforts are comparable to those tinkerers, facing the approach of the engineer, who would be one of the big Enterprise, they innovate sometimes unknowingly, because they are faced with a challenge, in fact, several surveys have shown that innovation is often a specific action in SMEs, it is usually conducted in a reactive way.

2.6.2. The Specialize Strategy

Specialization "is a simple approach that aims to focus the efforts of the firm in a market and on the same type of product. Often it is the only option available to a small business's success will be an advantage in terms of cost, or differentiation attributes or services". So, according to this definition the company that specializes to choose one of the generic strategies. Specialization according (Marmuse, 1996, p. 514) begins when the products and current missions allow the business to grow in line with its objectives.

Thus, according (Janssen, 2009, p. 61) "[...] A small company that seeks growth should focus on a limited number of products or services and fill gaps unoccupied by large companies", so, a strategy of specialization, may even go further than survival, and is used to achieve the goal of growth of small business.

2.6.3. Dependence Governance Strategy: Risk

Dependence is for SME strategic risk, which may occur when the small company carries the largest share of sales with a single customer, or by a single supplier. The head of the SME can take this problem by seeking an alliance, in fact, before setting the strategic alliance, it should highlight the concepts of each of cooperation and partnership, which seem to have the same meaning that the alliance.

• Cooperation is the fact performs an action together, treated as a common generic form of action which can range from simple exchanges of information at very advanced forms of collaboration between firms, including contractual terms. In this definition, integrates outsourcing, which is considered as (Montmorillon, 1997, p. 859) "one of the oldest form of business cooperation, it is a donor-order business, and another sub-contractor, where the second produces goods or services to the benefit of the first". And in this case the head of the SME needs to know to make himself indispensable to his client or principal.

• Partnership is practical collaborations that are most often implemented in a production chain, from upstream to downstream, by staging legally independent but economically dependent firms in the under a special agreement.

Regarding the alliance, according (Marchesnay, 1997, p. 126) it is a strategy that enables both companies (groups) or more to agree to set up joint units, taking a share of the capital in order to:

Develop new products, changing, and expertise; use the product produced in the site; strengthening distribution relationships, when some are suppliers of other distributors; penetrate protected international markets. Another reason that can lead companies to enter these agreements is for them to share the costs and development risks, which can not be handled alone.

For the team of (Strategor, 1997, p.140) strategic alliances can be defined as associations between several competitors, or potential competitors who choose to complete a project or specific activity by coordinating skills, means and resources rather than:

- To compete with each other on the activity;
- To merge them or to make acquisitions or disposals of activity.

So, according to this definition, alliances are essentially based on two principles, cooperation and a certain level of competition. In the context of SMEs, (Puthod, 1998, p. 95) said that this type of Firm, suffers from structural lack of resources, which may be the result of an alliance policy based on complementarily. Moreover, this strategy helps the company that wants to establish itself internationally and create the necessary means for this implementation, and this pathway may be a development accelerator.

2.6.4. The Diversification Strategy

Diversification than simply expands the business scope of the company. While specialization is based on the implementation of a set of scarce skills, diversification requires the use of a new set of skills required by the new competitive environment in which the Enterprise enters. According to (Janssen, 2009, p. 141) "Diversification is the preferred means of development for medium enterprises, where it should, however, be related to the existing activities of the Enterprise". So, for Ansoff I., among the causes that lead the SME to diversify is when the size of their business portfolio is insufficient to achieve their goal.

2.6.5. The Internationalization Strategy

In the context of globalization, economics and competition, companies are to internationalization strategies to greater or lesser degrees pushed. Exporting is the first stage of internationalization, which is the result of finding new customers overseas; in this case, it is advisable for smaller companies to master the local and national market before venturing into foreign markets.

In the second stage, the company will establish overseas sales office, where she can

continue with the installation or acquisition, alone or in joint venture, a production unit. In this approach, an SME can be supported by a large national company already set (portage or piggy back). The third stage is that of internationalization. The operating abroad units, whether subsidiaries or full, acquire a certain autonomy, and develop key functions needed to acquire appropriate skills distinctive to the host country. (Marchesnay, 1997, p. 34)

3. Empirical Study: Analyze and Interpretation of the Survey

We recall that our objective was to determine the characteristics of the strategic management of Algerian SMEs. In such an embodiment, and for reasons of proximity, the survey covered 43 companies, some located in the province of Mostaganem State, and some others in Oran state, where the majority of questionnaires were administered face to face. Analysis of the data collected is increased by the steps of:

- Encoding the data with the transformation of those qualitative to quantitative,
- Data entry using the SPSS 20.0,
- Tabulation of data in single or cross-tables,
- The tabulated data, which is the most important step in translating statistics to analyze the results.

3.1. Presentation of Data Response

The treatment of the data allowed us to synthesize the results in Table 1, which indicates that the rate of return is 51%, the number of rejected questionnaires is 2, and the number of usable questionnaires is 20, while, the response rate is 46%.

Table1. Data response of the SMEs sample

Distributed questionnaires number (1)	Number of questionnaires returned (2)	Return rate % (2 /1)	Number of questionnaires rejected (3)	Number of usable questionnaires (5)= (2-3)	Response rate % (6) = (5/ 1)
43	**22**	**51%**	**2**	**20**	**46%**

Source: Data of survey (2014), Authors contribution.

The return rate by sector of activity varies from one sector to another, where the highest (100%) response rate was recorded by the sector of building materials, ceramics and glass, and the industries textile industry, hosiery, garments, while the lowest rate (25%) relates to various industries. Noting that, the sector of the

industry, leather and footwear, the response rate is null.

3.2. The Sample: SMEs

The Figure 1 shows that SMEs are divided between the different legal statuses, including the LLC is the rating form (40% of SMEs), in the second place comes the EURL form with 30%, and on the SPA and individual account for each 15% of the sample.

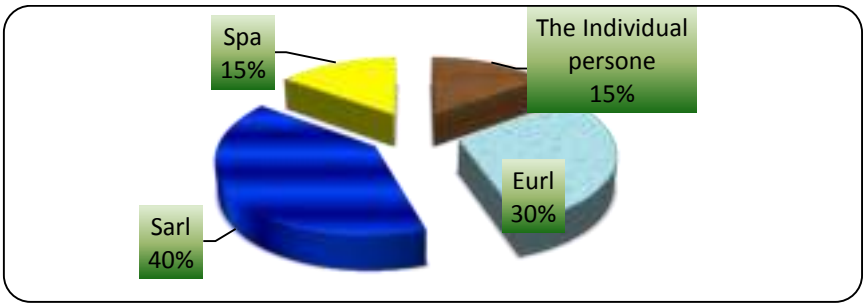


Figure 1. The legal form distribution of SMEs

Source: Data of survey (2014) (Authors contribution.)

According to Figure 2, the majority of SMEs in the sample just 60% meadows were created from 2001 because that date has been the promulgation of the Ordinance on the Investment Development (Ordinance No. 01 / 03 of 20/08/2001) and the framework law on the promotion of SMEs (law No. 01/18 of 12/12/2001), while 20% had between eleven and twenty four years, the remaining SMEs were created before 1988, the majority of family businesses.

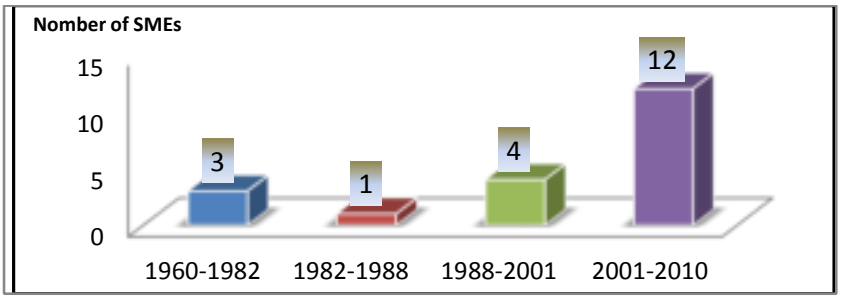


Figure 2. The distribution of SMEs by size

Source: Data of survey (2014) (Authors contribution)

In the next figure 3, our sample is composed of 8 Medium Enterprises, 6 Small Businesses and 6 Very Small Businesses.

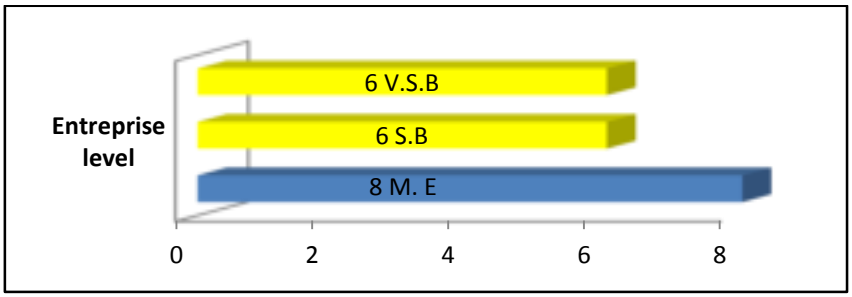


Figure 3. The distribution of SMEs by size

Source: Data of survey (2014) (Authors contribution)

3.3. Analyze and data discussion

The results of the survey were mined and analyzed to determine the characteristics of the Algerian SMEs. From Table 2, the majority of SMEs (17) responded that they have a strategy, while just 3 of them have no strategy.

Table 2. Cross-table strategy*aim

	Strategy			
		Yes	No	Total
The aim type	Write	7	0	7
	Inside the mind of the leader	10	3	13
	Total	17	3	20

Source: Data of survey (2014) (Authors contribution)

If a cross is made between planning objectives and a strategy, we find that among the companies that have a strategy that 07 firms (or 41.18%) make a plan, so their strategy is developed and explicit and that 10 companies (or 58.82% of companies with a strategy) their goals are in the heads of the leaders, their strategies are implicit and emerging over time.

3.3.1. The Strategic Vision: Governance strategy and Aim Type

Among the surveyed SMEs that are raised and the number of 17 and having a strategy, we note that these 16, a rate of 94% say they have a strategic vision against one said companies do not have a vision for this last, this seems contradictory because there is no strategy without vision.

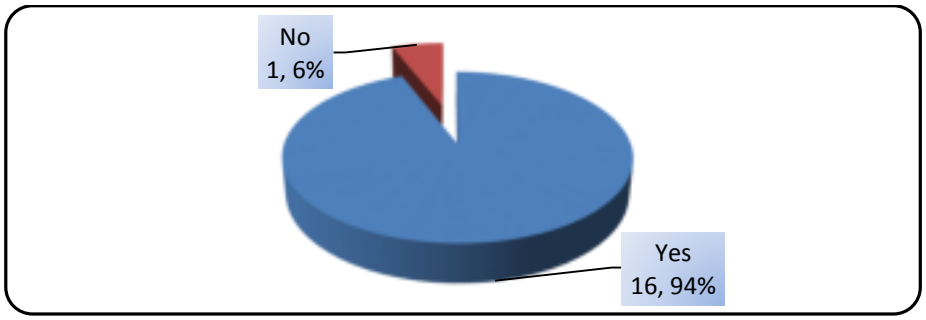


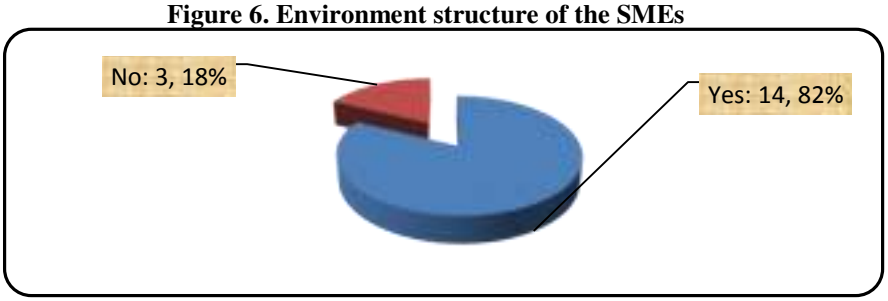
Figure 5. the strategic vision adopted by SMEs

Source: Data of survey (2014) (Authors contribution)

3.3.2. Environment Analyze

Concerning the analysis of the external environment, we find that 14 companies (82 companies with a strategy), are interested in analyzing their environment in terms of opportunities and threats, while 03 SMEs (18% of SMEs have responded strategy) does not take into account the external analysis.

Figure 6. Environment structure of the SMEs



Source: Data of survey (2014) (Authors contribution)

3.3.3. Translating Goal into Numbers

The SMEs that adopt strategies have to translate into measurable actions assigned objectives. According to Figure 7, this approach exists in 12 SMEs 71% against 5, a rate of 29%.

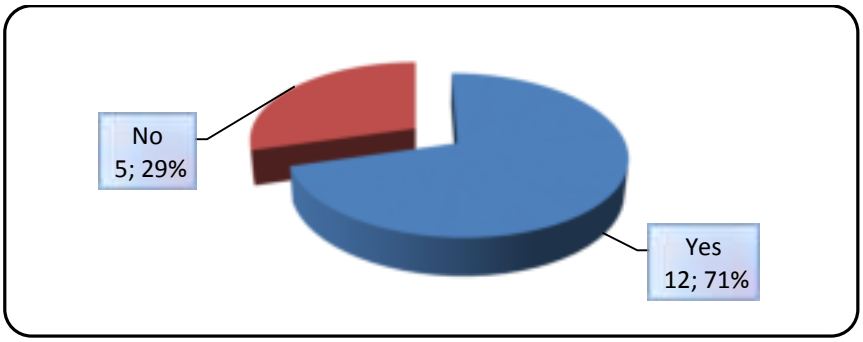


Figure 7. Environment structure of the SMEs

Source: Data of survey (2014) (Authors contribution)

3.3.4. The Formulation of the Strategy

The figure 8, show that 10 Enterprises of those respondents who adopted a strategy, do not set up, it is only in the heads of their leaders, while seven companies have deliberate strategies into plans.

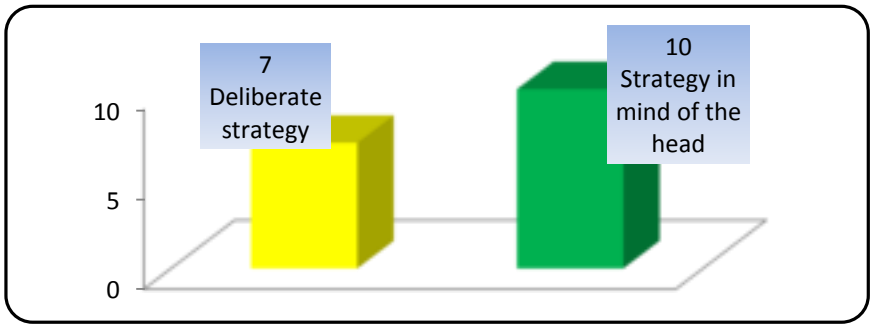


Figure 8. The SMEs Strategy formulating

Source: Data of survey (2014) (Authors contribution)

3.3.5. The Option Strategies

According to Figure 9, the majority of SME managers (82%) take the specialization strategies, while 18% of companies that responded have adopted a business strategy, trying to turn in another area of their different primary domain.

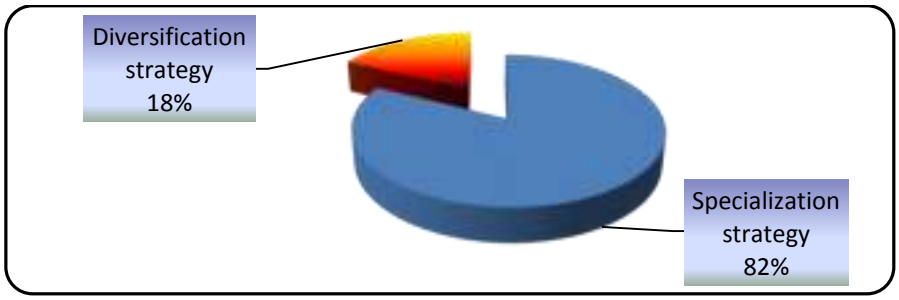


Figure 9. The SMEs option strategies

Source: Data of survey (2014) (Authors contribution)

3.3.6. The Alliance Strategy

Research collaboration and alliance with other partners from the same industry (cf. Figure 10), involves 9 SMEs (53% of companies surveyed), in contrast, 8 Enterprises (47%) are not interested in this type of strategy. So for these results, we can state that the Algerian SMEs have integrated new strategic practices including how external organization, based on changing business linkages to achieve their goals.

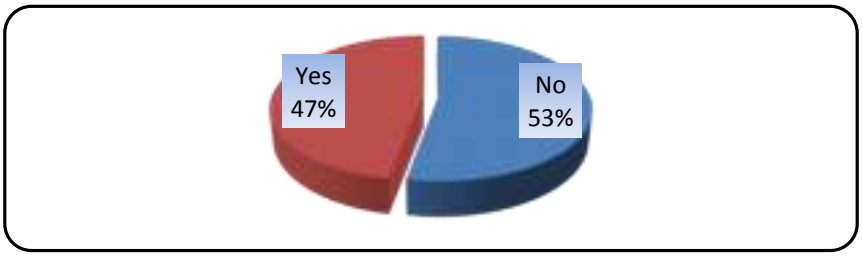


Figure10. The SMEs partnership and joint-venture

Source: Data of survey (2014) (Authors contribution)

3.3.7. The Internationalization Strategy

International development (cf. Figure 11) is difficult for Algerian SMEs because it is conditioned by internal and external (Toubache, 1998, p. 98) factors, but despite this we find that 29% of SMEs that responded have developed a strategy, adopt internationalization strategies, while 71% of SMEs do not choose their management practices, where the export is the most practiced strategy.

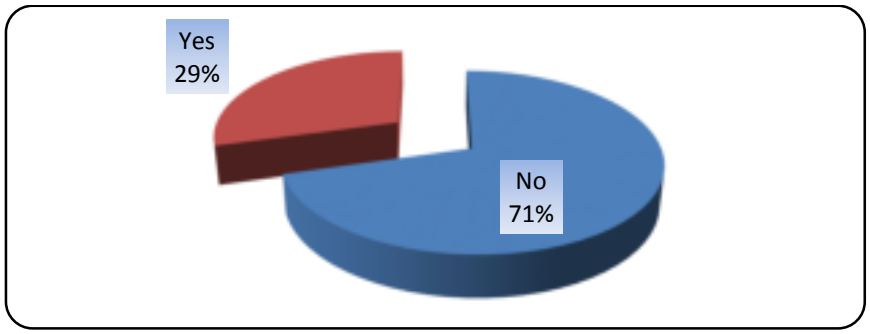


Figure 11. The SMEs internationalization strategy adopted

3.3.8. The Innovation Strategy and ICT

The following figure 12 shows that the majority of companies 59% are trying to innovate frequently, and by the creation of new products or the development of processes for production and marketing, while 41% of SMEs do not.

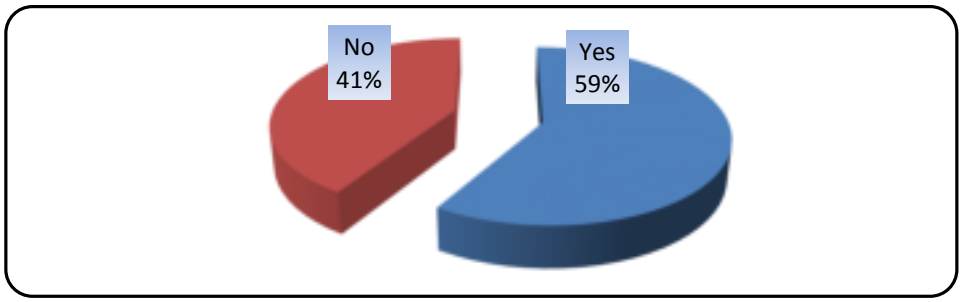


Figure 12. The SMEs innovation strategy adopted

But examination of responding companies have innovated indicates those only four Enterprises or 40% of those respondents who have an innovation strategy based on a research and development.

3.3.9. Governance Strategy: Typology

Through the different results we were able to develop a typology of SMEs (*cf.* Table 3) depending on the characteristics of their strategy, strategic approach, and their most strategic option adopted.

Table 3. Typology of the SMEs and management strategy

Type	Size	Strategic system	Strategic method	Option strategy	Observations
Type 1	50% of SMEs in the sample	Incluses all strategic pillars	Mostly implicit (60% of such companies).	80% adopted the strategy of specialization	20% adopte new management techniques
Type 2	35% of SMEs in the sample	lack of a strategic pillar or more	Mostly implicit (71.43% of Firms)	71.43% of SMEs. adopt the strategy of specialization	28.57% practice new management techniques

Source: Survey data. (Authors contribution)

4. Conclusion

The purpose of our research was to examine the strategic management in SMEs, under the theme "Strategies for SMEs: the case of SMEs in western Algeria." Indeed, the empirical study of a sample of 20 SMEs, found that generally, the strategy of the Algerian SMEs, is between an implicit strategy and implicit strategic practices, development of the company relies mainly on the strategy of specialization. The various analyzes of the survey yielded the following results:

- 85%, the majority of companies respond that they have a strategy, but in reality 50% of companies in the sample have a strategy.
- 60% of companies actually have a strategy, it is implicit and emerges over time, because the leaders of these companies do not use strategic planning, but their strategy is in their heads.
- The strategic choice shows that the most widely adopted by SMEs in our sample policy option is the strategy of specialization, which is 80%, compared with the diversification strategy that is adopted by only 20% of SMEs in sample.

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Empirical Study on Accounting and Tax Differences in the Case of an Economic Entity

Valeriu Laurenţiu Onose[1], Ecaterina Necşulescu[2]

Abstract: Often enough, the definitions in tax regulations are not unique, and they can differ even within the fiscal legislation from one title to another. Thus the accounting regulations define the net turnover tax code, but at the same time it brings different definitions of turnover in terms of added value. The turnover for professionals in accounting is an indicator used to represent the value and the synthetic of the achieved sales by an economic unit during a certain period or a business segment. Also the fiscal elements come to generate a new identity of the taxable result, totally different from the accounting result. There are found clear value differences between accounting and tax perspective on the outcome in terms of operating profit and expenses. These differences at the level of operating costs continue to affect key indicators of the structure of the profit account and loss until determining the gross profit, which differs in the accounting vision compared to the gross profit in the fiscal vision.

Keywords: tax turnover; accounting turnover; gross profit in the accounting vision; gross profit in the taxing vision; taxation

1. The Tax and Accounting Differences at the Level of Terminology Regarding the Turnover

Professional accountants understand the concept of turnover as defined by the statutory accounting rules (Finance Minister's Orders 3055/2009), and often they overlook that tax regulations are calling on other specific definitions for certain items already defined in the specialized literature and in the accounting legislation. The accounting regulation defines the net turnover comprising "proceeds from the sale of products and providing services falling under the current activity of the entity, after deducting the commercial discounts and value added tax and other taxes directly related to business turnover" (Finance Minister's Orders 3055/2009, pt. 33, paragraph 1).

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Meanwhile the Tax Code brings different definitions of turnover in terms of VAT:

1. The turnover as a price ceiling to the establishment of the special regime of exemption for small enterprises:

According to the Tax Code, the turnover comprises: the total value, excluding tax (VAT) of supplies of goods and services made by the taxable person during a calendar year, taxable or, where appropriate, they would be taxable if the operations were not carried out by a small company, representing the intra-community transactions of goods, providing intra-community services of other deliveries for which the place is considered to be abroad, transactions exempt from the VAT and those exempt without deductibility provided art. 141, paragraph (2), letters a), b), e) and f) if they are not ancillary to the main business, except the following:

a) supply of tangible or intangible fixed assets, as defined by the tax code;

b) intra-Community supplies of new means of transport, exempt under the tax code.

2. The turnover as a price ceiling for the establishment of the fiscal period includes taxable transactions and/or exempted from the VAT and/or non-taxable in Romania, but granting the right to be deductible.

For all the above two price ceilings, according to the Tax Code, the entities will be considered in the calculation of turnover including receipts or invoices for unearned advances and other bills issued before the date of delivery / provision.

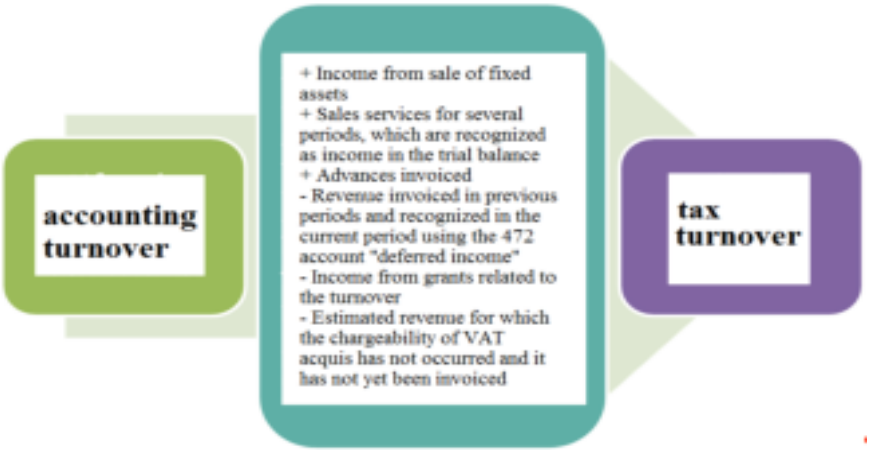

Figure 1. The transition from tax turnover to the accounting turnover

At the same time, the tax turnover includes some items that are not included in the accounting turnover such as the sales of assets, sales of services for several periods, which are recognized as deferred income in the trial balance, advances billed but, at

the same time, it excludes certain items from the accounting turnover namely: the revenue invoiced in previous periods and recognized in the current period using the 472 account "deferred income", income from subsidies related to the turnover, estimated revenues for which the chargeability of VAT has not been held and has not been yet invoiced.

2. Accounting and Tax Perspective on Gross Operating Result

The influence of taxation on accounting often leads to distortion of the economic content of the presented indicators. A good example is the comparative analysis of presenting the key indicators of income and expenses in the Form 20 of financial statements and those presented in Statement 101 - Income Tax Statement, and more specifically, the content of operating expenses from the accounting point of view versus the content of expenses operating expenses from the taxing point of view. If in the case of profit and loss account, operating expenses are "uncovered" by fiscal influences, we find that the instructions for completing the form 101 "Statement regarding income tax", where it specifies at line 2 "operating expenses" to complete also the "expense with income tax."

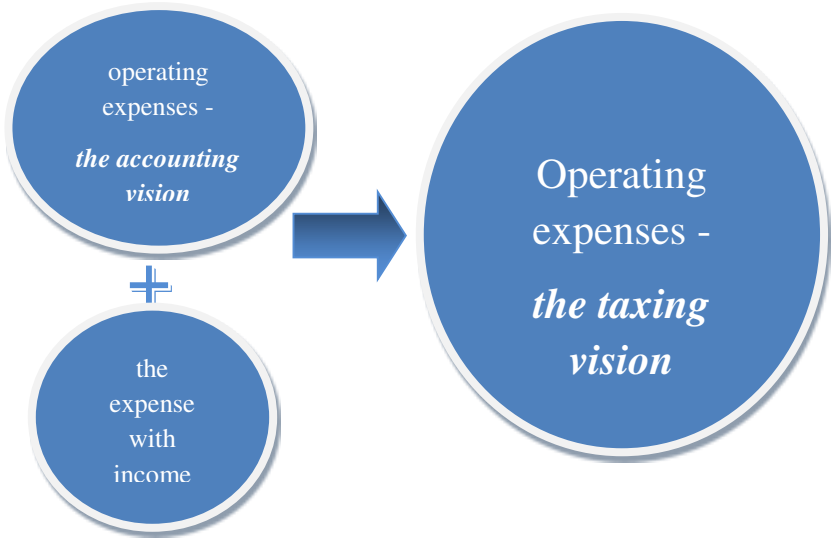

Figure 2. Operating expenses - accounting vision vs. tax vision
Source: Authors' processing

In the case of our company, we find clear value differences between the two visions in the following table showing the operating result, depending on the interpretation, the accounting or taxing.

Table 1. Profit and loss according to accounting and tax vision

Elements of the CPP according to the accounting vision	2007	2008	2009	2010	2011	2012	2013
Operating income	669.611	1.077.801	1.540.995	3.689.860	2.918.780	1.513.647	1.691.644
Operating expenses	486.826	673.901	1.061.600	1.598.350	2.139.524	1.728.611	1.528.727
Operational result sheet (RexC)	*182.785*	*403.900*	*479.395*	*2.091.511*	*779.256*	*-214.964*	*162.917*
Elements of the CPP as tax vision	2007	2008	2009	2010	2011	2012	2013
Operating income	669.611	1.077.801	1.540.995	3.689.860	2.918.780	1.513.647	1.691.644
Operating expenses (including income tax expense, according to the statement 101)	513.685	735.746	1.129.167	1.911.926	2.247.475	1.732.381	1.528.727
Tax Operational result (RexF)	*155.926*	*342.056*	*411.829*	*1.777.934*	*671.305*	*-218.734*	*162.917*

The differentiation between tax and accounting vision is evident in the considered period, which is more extensive in the period 2009 - 2011, following a decrease, and towards the end of the period, an equalization in terms of value of the two types of operating results.

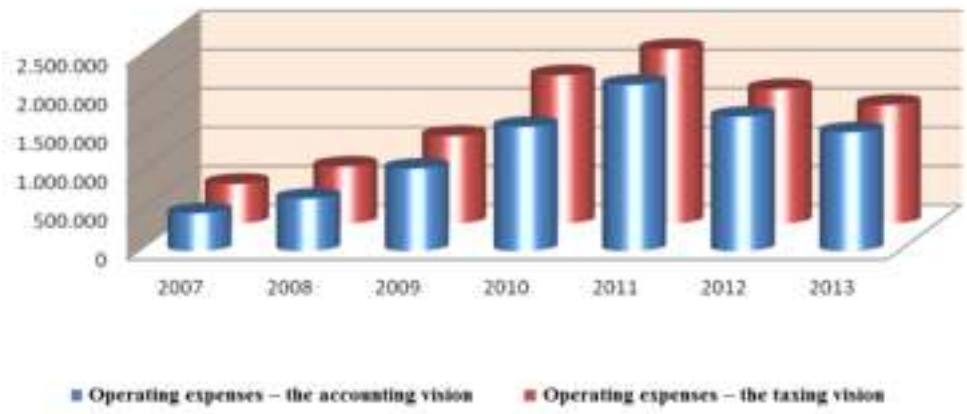

Figure 3. The Evolution of operating costs in the two visions

This differentiation is observed more clearly from the chart below, the cause and magnitude of different values of this indicator is given just by the income tax expense.

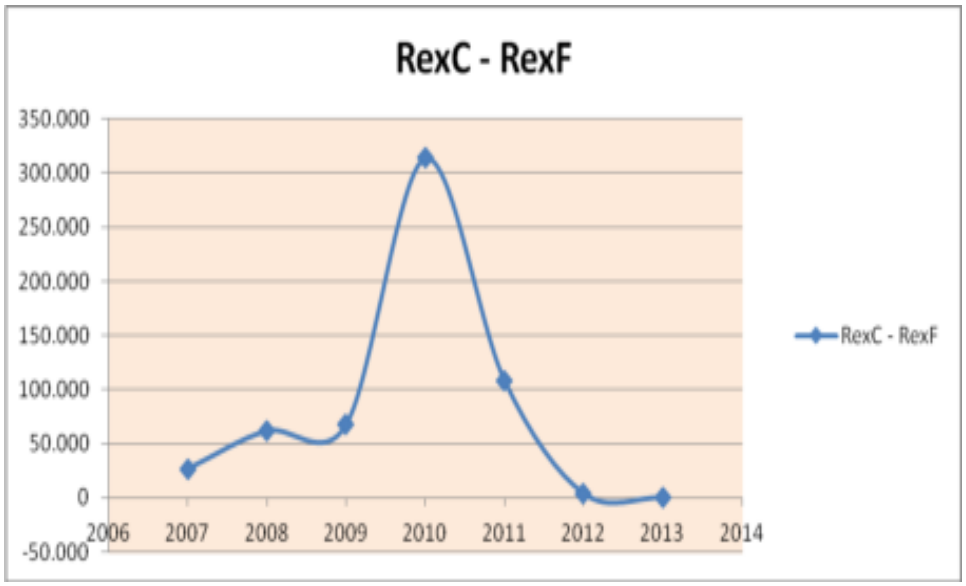

Figure 4. The Evolution of the difference between the tax and accounting operating income

It results that given the magnitude of the difference between the calculation of the operating result in fiscal and accounting vision, the result is kept downstream to determine gross income, both in the tax and accounting vision. The following figure has highlighted the developments in the period under review two visions.

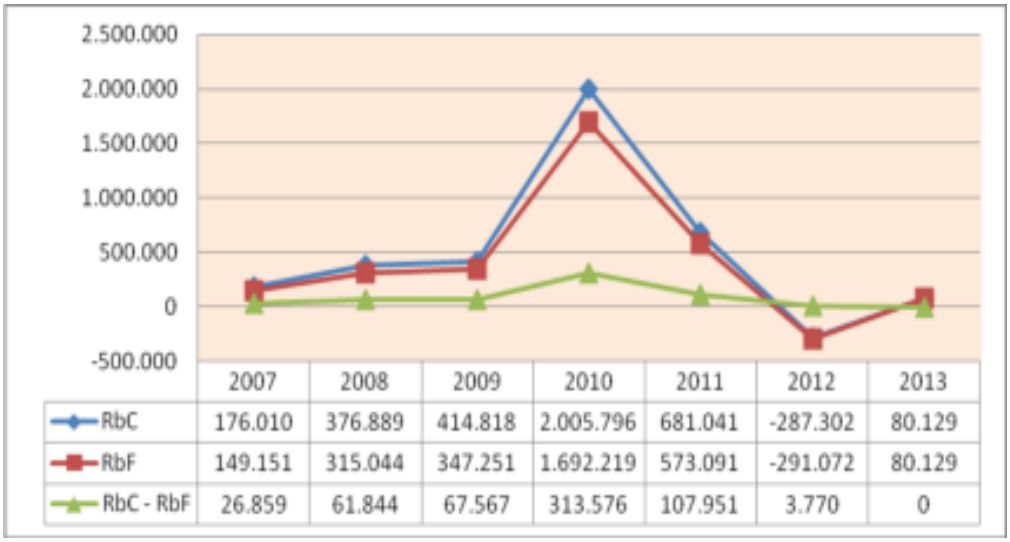

Figure 5. The evolution of gross accounting vs taxing operating outcome

From the above figure it results that the magnitude of the variation is given by the value of tax expenditures.

Table 2. The determination in the accounting and tax vision of gross operating outcome

No.	Elements of Profit and Loss Account		2007	2008	2009	2010	2011	2012	2013
1	Accounting vision	Operating income	669.611	1.077.801	1.540.995	3.689.860	2.918.780	1.513.647	1.691.644
2		Operating expenses	486.826	673.901	1.061.600	1.598.350	2.139.524	1.728.611	1.528.727
3		Operational result (rd.1 - rd.2)	*182.785*	*403.900*	*479.395*	*2.091.511*	*779.256*	*-214.964*	*162.917*
4		financial income	2.781	786	26.172	66.471	56.639	129.073	106.742
5		financial expenses	9.556	27.797	90.749	152.185	154.854	201.411	189.530
6		Financial result (Rd.4 - rd.5)	*-6.775*	*-27.011*	*-64.577*	*-85.715*	*-98.215*	*-72.338*	*-82.788*
7		extraordinary income	0	0	0	0	0	0	0

8		extraordinary expenses	0	0	0	0	0	0	0
9		Extraordinary Result (rd.7-rd.8)	*0*	*0*	*0*	*0*	*0*	*0*	*0*
10		Gross (rd.3 + R d-6 + rd.9) (RBC)	*176.010*	*376.889*	*414.818*	*2.005.796*	*681.041*	*-287.302*	*80.129*
11	Tax vision	Operating income	669.611	1.077.801	1.540.995	3.689.860	2.918.780	1.513.647	1.691.644
12		Operating expenses	513.685	735.746	1.129.167	1.911.926	2.247.475	1.732.381	1.528.727
13		Operational result (rd.11 - rd.12)	*155.926*	*342.056*	*411.829*	*1.777.934*	*671.305*	*-218.734*	*162.917*
14		financial income	2.781	786	26.172	66.471	56.639	129.073	106.742
15		financial expenses	9.556	27.797	90.749	152.185	154.854	201.411	189.530
16		Financial result (rd.14 - rd.15)	*-6.775*	*-27.011*	*-64.577*	*-85.715*	*-98.215*	*-72.338*	*-82.788*
17		extraordinary income	0	0	0	0	0	0	0
18		extraordinary expenses	0	0	0	0	0	0	0
19		Extraordinary Result (rd.17-rd.18)	*0*	*0*	*0*	*0*	*0*	*0*	*0*
20		Gross (rd.13 + rd.16 + rd.19) (RBF)	*149.151*	*315.044*	*347.251*	*1.692.219*	*573.091*	*-291.072*	*80.129*
RbC - RbF			***26.859***	***61.844***	***67.567***	***313.576***	***107.951***	***3.770***	***0***

The above table presents the differences between the gross operating result in the accounting and tax vision, using the indicators of the analyzed company in the period 2007-2013.

3. The Accounting Result versus Tax Result

In the common language of professional accountants, the notion of *profit* is identified with the one of *outcome*, even if at the semantic level it has a speculative acceptation, while the outcome cannot be viewed only as a consequence of an action or accumulation of actions under the rule of causality. The concept of outcome is far more generous, not only in terms of the specifics of economic activities, involving investment followed by the outcome, but also because it accepts and it incorporates the alternative of profit, i.e. the loss.

A secular approach of terminologies with which the taxing and accounting are operating, would let to understand that the tax is calculated by applying a percentage

to the positive gross result of the economic entity. But the practice is completely different.

The path from the accounting outcome based on the fiscal imposition is more complex, full of challenges arising from the differences that arise between the accounting policy of the company and state fiscal philosophy.

For the purposes of determining the taxable base, the Romanian Tax Code comes with a wide range of adjustments and influences that bring the individual components included in the income statement, plus a series of tax incentives, according to the state fiscal policy.

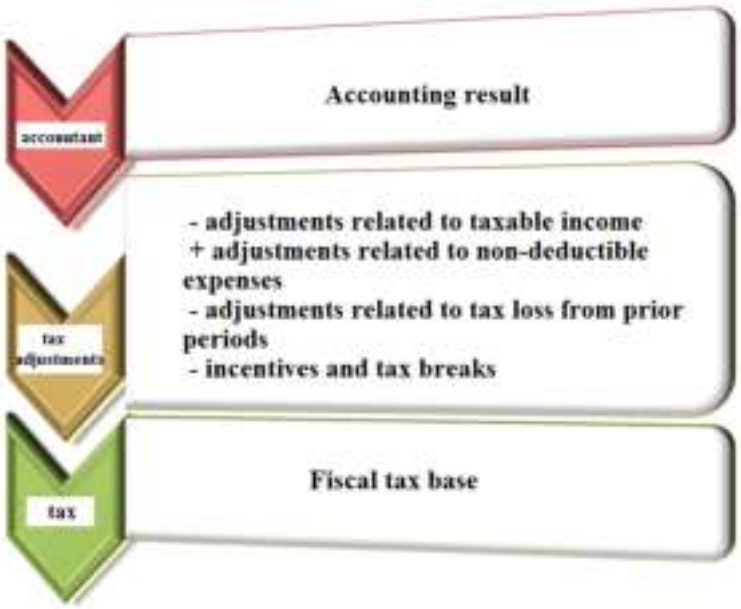

Figure 6. Switching from gross accounting profit at the fiscal tax base

In the case of the analyzed economic unit, switching from gross accounting profit to the taxable profit is shown in the table below:

Table 3. Determination of the basis of taxation

No	Indicators		2007	2008	2009	2010	2011	2012	2013
1	Determining the accounting outcome	Operating income	669.611	1.077.801	1.540.995	3.689.860	2.918.780	1.513.647	1.691.644
2		Operating expenses	486.826	673.901	1.061.600	1.598.350	2.139.524	1.728.611	1.528.727
3		Operational outcome (rd.1 - rd.2)	*182.785*	*403.900*	*479.395*	*2.091.511*	*779.256*	*-214.964*	*162.917*
4		financial income	2.781	786	26.172	66.471	56.639	129.073	106.742
5		financial expenses	9.556	27.797	90.749	152.185	154.854	201.411	189.530
6		Financial result (Rd.4 - rd.5)	*-6.775*	*-27.011*	*-64.577*	*-85.715*	*-98.215*	*-72.338*	*-82.788*
7		extraordinary income	0	0	0	0	0	0	0
8		extraordinary expenses	0	0	0	0	0	0	0
9		Extraordinary outcome (rd.7-rd.8)	*0*	*0*	*0*	*0*	*0*	*0*	*0*
10		Gross accounting outcome (rd.3 + R d-6 + rd.9)	*176.010*	*376.889*	*414.818*	*2.005.796*	*681.041*	*-287.302*	*80.129*
11	Determining the taxing outcome	Adjustments related to taxable income	0	0	0	54.060	9.472	0	80.129
12		Adjustments related to non-deductible expenses	6.327	9.636	7.476	8.114	3.124	310.864	0
13		Adjustments related to prior period tax loss	14.468	0	0	0	0	0	0
14		Incentives and tax breaks	0	0	0	0	0	0	0
15		The basis of taxation (rd.10-rd.11 + rd.12-rd.13-rd.14)	*167.869*	*386.525*	*422.294*	*1.959.850*	*674.694*	*23.563*	*0*
16		Corporation tax (rd.15 x 16%)	*26.859*	*61.844*	*67.567*	*313.576*	*107.951*	*3.770*	*0*

It appears that in every year from the analyzed period, there were adjustments that led to the value delineation of the gross accounting outcome based on taxation. A graphical representation of this development clearly highlights these differences.

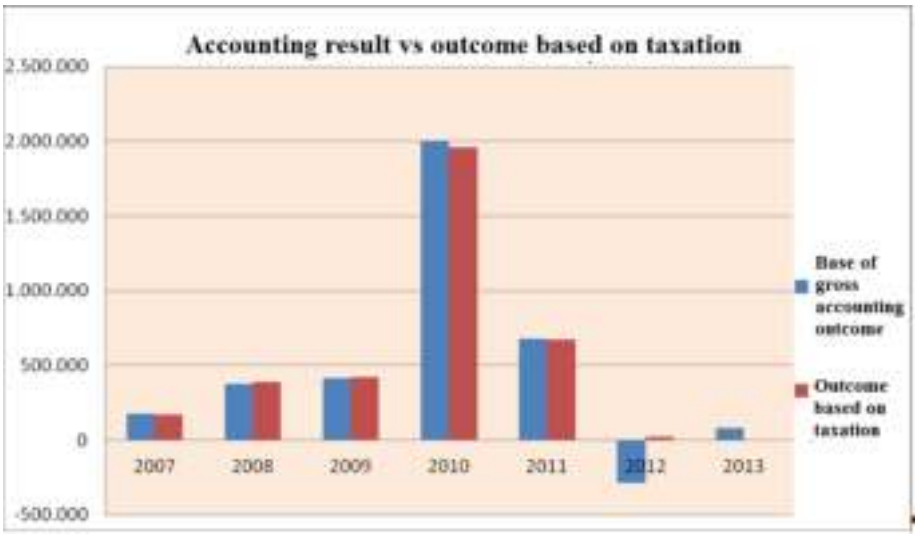

Figure 7. The evolution of the accounting result versus outcome based on taxation

Linked to the developments described in the previous figure, we note that during 2007 - 2013, the adjustments were relatively small, except 2012, when the unit recorded in the accounts a large volume of expenditure considered non-deductible for tax purposes.

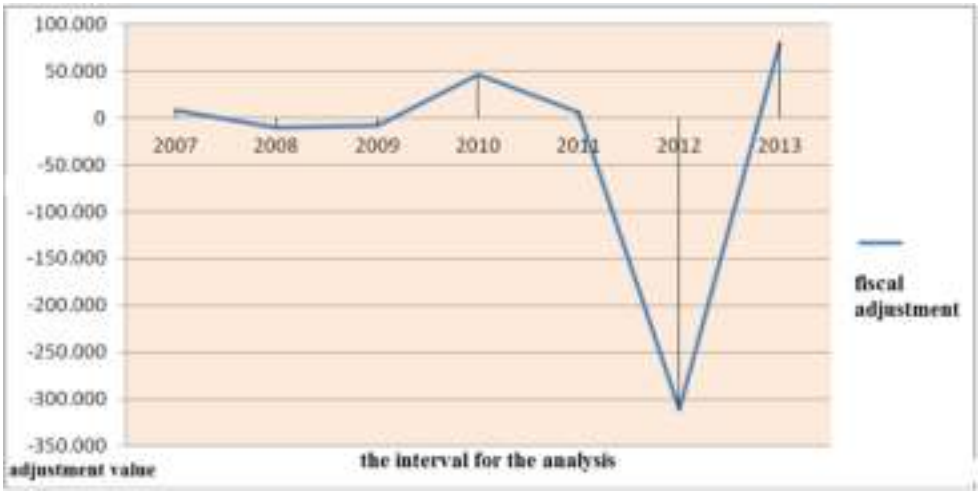

Figure 8. The Evolution of fiscal adjustments in 2007-2013

The magnitude of these influences on the gross accounting outcome is revealed by the evolution of the influence index of fiscal adjustments and the gross accounting outcome (calculated as their ratio).

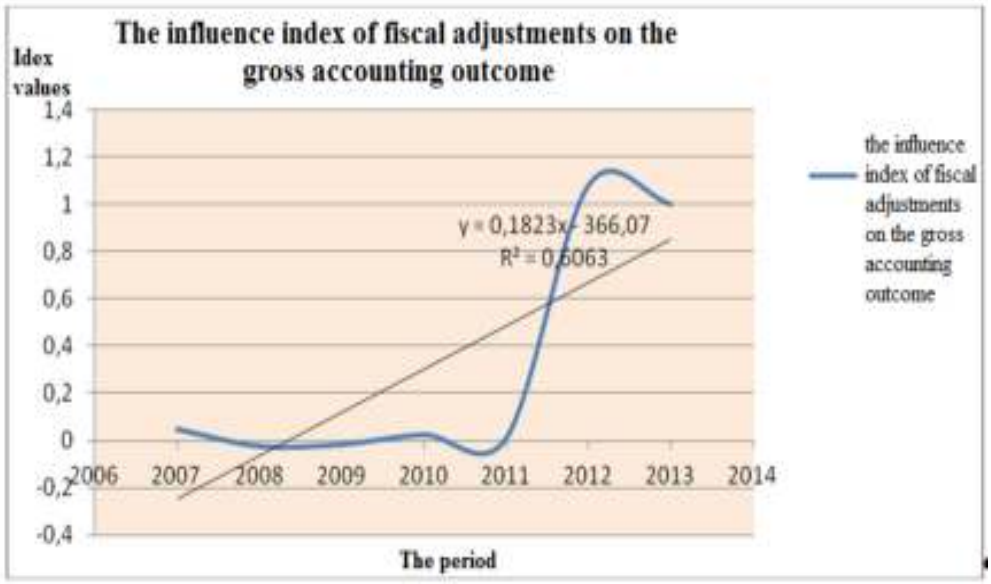

Figure 9. The evolution of the influence index of fiscal adjustments on the gross accounting outcome

We find that the intensity of this index increases towards the end of the analyzed time period, due to the leverage resulting from diminishing the gross accounting outcome compared with the average increase of fiscal adjustments in this period.

A comparative analysis of the influences of fiscal adjustments on gross operation outcome, i.e. on the operating income that confirms the same trend of development as in the case of influence index of fiscal adjustments on the gross accounting outcome, but of a lesser magnitude, as seen in the following figure:

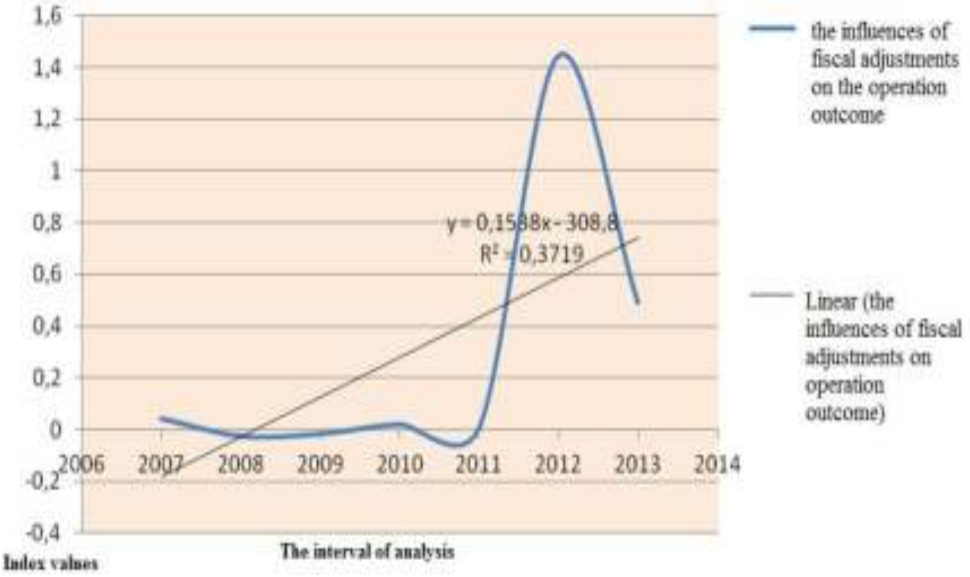

Figure 10. The Evolution of the influence index of fiscal adjustments on the operating outcome

Another interesting analysis is the comparison of fiscal adjustments influences on operating revenues and expenses, as defined in the accounting terms, given that their economic substance is affected by these adjustments.

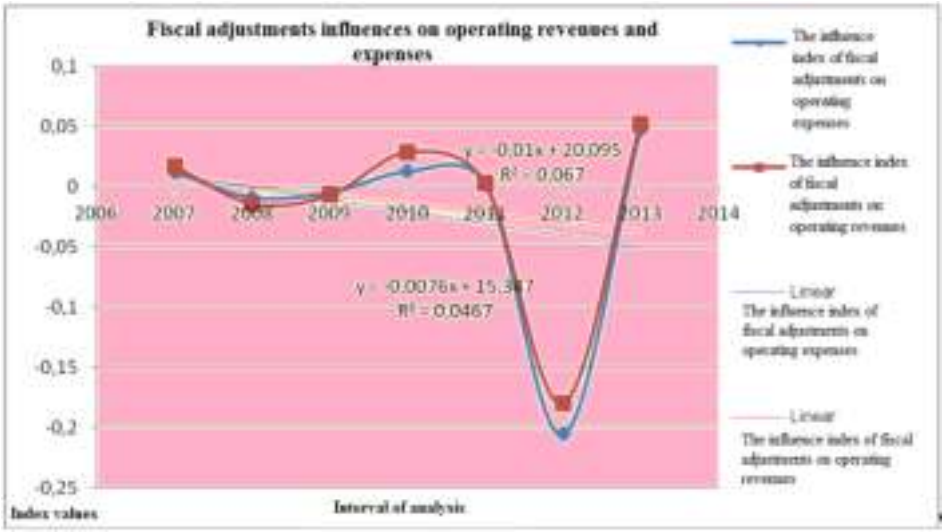

Figure 11. Fiscal adjustments on operating revenues and expenses

Graphically, we find that fiscal adjustments are as significant as the operating revenues and on operating expenses, with a slight superior tendency in terms of operating revenues.

4. Conclusions

As a conclusion to the matter described above, although complementary in terms of the origin or economic transaction, accounting and taxation operate with a set of indicators, in terms of semantics, similar but distinct in terms of scope and definition. The values that we derive from accounting reports often do not have the same valence and fiscal significance, having to make recourse to a complex set of adjustments in order to connect them.

We find that accounting and tax legislation define the turnover in their own way and in everyday activity and everyday language of professional accountants, the two definitions coexist, which can generate high risks of misunderstanding the turnover. The domain and context in which it makes use of this concept should be clearly delimited, in order to avoid the serious consequences that may arise at the application level; in the fiscal or accounting context any errors of interpretation can generate increases and penalties, if it is identified during a tax audit.

No matter how useful it would be in taxing terms an expenditure classification, in the sense of including the income tax with profit taxing in operating expenses of the company, such an approach could create confusion in the subsequent analysis of the position indicators and the performance of an economic unit.

We can say, in our view, that the concept of *account of results* is more appropriate for the *account of profit and loss*, given that both valences account cannot be, positive or negative.

The involvement of taxation in this semantic divergence has the gift of granting a high percentage to the concept of profit and loss account, in the light of the objective pursued by the tax: *profit tax*. This emphasis is manifested only due to the tax titles, although the real fiscal objective is to determine the basis of taxation, and consequently the profit tax.

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