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ŒCONOMICA

Determination of Unconditional Extremes of the Functions of Fundamental Symmetric Polynomials

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Abstract: The paper demonstrates how the unconditional extremes of the functions of fundamental symmetric polynomials, in the case of the positivity of variables, are obtained either by canceling the partial derivatives in relation to the polynomials, or by the equality of all variables.

Keywords: extreme; symmetric polynomials; inequality

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1. Introduction

Let $x_1,\dots,x_n \in \mathbf{R}$, $n \in \mathbf{N}$, $n \geq 2$ and fundamental symmetric polynomials of order k:

$$S_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \dots x_{i_k},\ k = \overline{1,n}.$$

We'll assume that $x_1,\dots,x_n \neq 0$, otherwise considering only non-zero variables with n corresponding to their number.

Let's also note, $S_0=1$, $S_{-p}=0$, $S_{n+p}=0$ $\forall p \geq 1$ and $S_k^{p_1,\dots,p_s}$, $k=\overline{1,n}$, $1 \leq p_1 \neq \cdots \neq p_s \leq n$, fundamental symmetric polynomials of order k in $x_1,\dots,\widehat{x_{p_1}},\dots,\widehat{x_{p_s}}\dots,x_n$ where ^ means that the factor is missing and, as above, $S_0^{p_1,\dots,p_s}=1$, $S_{-p}^{p_1,\dots,p_s}=0$ $\forall p \geq 1$, $S_p^{p_1,\dots,p_s}=0$ if $p>n-s$.

We have:

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$S_k^{p_1,\dots,p_s} = \sum_{1\le i_1<\dots<\widehat{p_1}<\dots<\widehat{p_s}<\dots i_k\le n} x_{i_1}\dots\widehat{x_{p_1}}\dots\widehat{x_{p_s}}\dots x_{i_k}=x_j S_{k-1}^{j,p_1,\dots,p_s} + S_k^{j,p_1,\dots,p_s}$, $j\neq p_1\neq\dots\neq p_s$.

$S_k^{i,p_1,\dots,p_s} - S_k^{j,p_1,\dots,p_s} = x_j S_{k-1}^{j,i,p_1,\dots,p_s} + S_k^{j,i,p_1,\dots,p_s} - x_i S_{k-1}^{j,i,p_1,\dots,p_s} - S_k^{j,i,p_1,\dots,p_s} = (x_j - x_i) S_{k-1}^{j,i,p_1,\dots,p_s}$, $i\neq j\neq p_1\neq\dots\neq p_s$.

from where:

$\frac{\partial S_k}{\partial x_i}=\sum_{1\le i_1<\dots<\hat{\imath}<\dots<i_k\le n} x_{i_1}\dots\widehat{x_i}\dots x_{i_k}=S_{k-1}^i$, k=$\overline{0,n}$, i=$\overline{1,n}$,

$\frac{\partial S_k^{p_1,\dots,p_s}}{\partial x_i}=S_{k-1}^{i,p_1,\dots,p_s}$.

2. Main Theorem

Let $x_1,\dots,x_n\in\mathbf{R}$, $n\in\mathbf{N}$, n≥

Let be a function of fundamental symmetric polynomials: f:$\mathbf{R}^n\rightarrow\mathbf{R}$, $f(S_1,\dots,S_n)$.

We have: $\frac{\partial f}{\partial x_i}=\sum_{k=1}^n \frac{\partial f}{\partial S_k}\frac{\partial S_k}{\partial x_i}=\sum_{k=1}^n \frac{\partial f}{\partial S_k} S_{k-1}^i$, and also:

$\frac{\partial^2 f}{\partial x_i\partial x_j}=\frac{\partial}{\partial x_j}\sum_{k=1}^n \frac{\partial f}{\partial S_k} S_{k-1}^i=\sum_{k,s=1}^n \frac{\partial^2 f}{\partial S_k\partial S_s} S_{k-1}^i S_{s-1}^j + \sum_{k=1}^n \frac{\partial f}{\partial S_k} S_{k-2}^{i,j}$ if i≠j and:

$\frac{\partial^2 f}{\partial x_i^2}=\frac{\partial}{\partial x_i}\sum_{k=1}^n \frac{\partial f}{\partial S_k} S_{k-1}^i=\sum_{k,s=1}^n \frac{\partial^2 f}{\partial S_k\partial S_s} S_{k-1}^i S_{s-1}^i$.

From the above formulas, we obtain:

$$d^2 f = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i\partial x_j} dx_i dx_j$$
$$= \sum_{i,j=1}^n \left(\sum_{k,s=1}^n \frac{\partial^2 f}{\partial S_k\partial S_s} S_{k-1}^i S_{s-1}^j + (1-\delta_{ij})\sum_{k=1}^n \frac{\partial f}{\partial S_k} S_{k-2}^{i,j}\right) dx_i dx_j$$

where δ_{ij}=1 if i=j and 0 if i≠j is the Kronecker's symbol.

We aim to determine the local extremes of the function f.

To determine the stationary points, the characteristic system becomes:

$\frac{\partial f}{\partial x_i}=\sum_{k=1}^n \frac{\partial f}{\partial S_k} S_{k-1}^i$=0, i=$\overline{1,n}$ or, deployed:

$$\begin{cases} \frac{\partial f}{\partial S_1} + \frac{\partial f}{\partial S_2} S_1^1 + \cdots + \frac{\partial f}{\partial S_n} S_{n-1}^1 = 0 \\ \cdots \\ \frac{\partial f}{\partial S_1} + \frac{\partial f}{\partial S_2} S_1^n + \cdots + \frac{\partial f}{\partial S_n} S_{n-1}^n = 0 \end{cases}$$

If all partial derivatives $\frac{\partial f}{\partial S_k}$=0 at one point $(x_1^0, \ldots, x_n^0)$ then the system is checked. In this case:

$$d^2 f = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j = \sum_{i,j=1}^{n} \left(\sum_{k,s=1}^{n} \frac{\partial^2 f}{\partial S_k \partial S_s} S_{k-1}^i S_{s-1}^j \right) dx_i dx_j$$

It is known that if d^2f is positively defined then the point is of local minimum, if it is negatively defined then the point is of local maximum, and if it is semi-defined the point is not extremely local.

If at least one of the partial derivatives $\frac{\partial f}{\partial S_k}$≠0 then the system does not support only the null solution, therefore the determinant of the system is null:

$$\begin{vmatrix} 1 & S_1^1 \ldots & S_{n-1}^1 \\ 1 & S_1^2 \ldots & S_{n-1}^2 \\ \ldots & \ldots\ldots & \ldots \\ 1 & S_1^n \ldots & S_{n-1}^n \end{vmatrix} = 0.$$

Subtracting the first line from the others, results in:

$$0 = \begin{vmatrix} 1 & S_1^1 \ldots & S_{n-1}^1 \\ 1 & S_1^2 \ldots & S_{n-1}^2 \\ \ldots & \ldots\ldots & \ldots \\ 1 & S_1^n \ldots & S_{n-1}^n \end{vmatrix} = \begin{vmatrix} 1 & S_1^1 & \ldots & S_{n-1}^1 \\ 0 & (x_1 - x_2) S_0^{1,2} & \ldots & (x_1 - x_2) S_{n-2}^{1,2} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & (x_1 - x_n) S_0^{1,n} & \ldots & (x_1 - x_n) S_{n-2}^{1,n} \end{vmatrix} =$$

$$\prod_{j=2}^{n} (x_1 - x_j) \begin{vmatrix} 1 & S_1^{1,2} \ldots & S_{n-2}^{1,2} \\ 1 & S_1^{1,3} \ldots & S_{n-2}^{1,3} \\ \ldots & \ldots\ldots & \ldots \\ 1 & S_1^{1,n} \ldots & S_{n-2}^{1,n} \end{vmatrix} = \ldots = \prod_{1 \le i < j \le n} (x_i - x_j).$$

It follows from this expression that at least two variables must be equal.

Suppose, after a possible renumbering, that $x_1 = x_2$.

How, in this case: $S_k^1 = S_k^2$, k=$\overline{1,n}$ the second equation of the characteristic system is identical to the first, so we have:

$$\begin{cases} \frac{\partial f}{\partial S_1} + \frac{\partial f}{\partial S_2} S_1^2 + \cdots + \frac{\partial f}{\partial S_n} S_{n-1}^2 = 0 \\ \cdots \\ \frac{\partial f}{\partial S_1} + \frac{\partial f}{\partial S_2} S_1^n + \cdots + \frac{\partial f}{\partial S_n} S_{n-1}^n = 0 \end{cases}$$

In this case:

$$\frac{\frac{\partial f}{\partial S_1}}{\begin{vmatrix} S_1^2 S_2^2 \cdots S_{n-2}^2 S_{n-1}^2 \\ \cdots \cdots \cdots \cdots \cdots \\ S_1^n S_2^n \cdots S_{n-2}^n S_{n-1}^n \end{vmatrix}} = \frac{\frac{\partial f}{\partial S_2}}{\begin{vmatrix} S_2^2 S_3^2 \cdots S_{n-1}^2 \, 1 \\ \cdots \cdots \cdots \cdots \cdots \\ S_2^n S_3^n \cdots S_{n-1}^n \, 1 \end{vmatrix}} = \frac{\frac{\partial f}{\partial S_3}}{\begin{vmatrix} S_3^2 S_4^2 \cdots 1 \, S_1^2 \\ \cdots \cdots \cdots \cdots \cdots \\ S_3^n S_4^n \cdots 1 \, S_1^n \end{vmatrix}} = \cdots$$

$$= \frac{\frac{\partial f}{\partial S_n}}{\begin{vmatrix} 1 \, S_1^2 \cdots S_{n-3}^2 S_{n-2}^2 \\ \cdots \cdots \cdots \cdots \cdots \\ 1 \, S_1^n \cdots S_{n-3}^n S_{n-2}^n \end{vmatrix}}$$

which is equivalent to:

$$\frac{\frac{\partial f}{\partial S_1}}{x_1^{n-1} \prod_{2\le i<j\le n}(x_i - x_j)} = \cdots = \frac{\frac{\partial f}{\partial S_k}}{x_1^{n-k} \prod_{2\le i<j\le n}(x_i - x_j)} = \cdots$$

$$= \frac{\frac{\partial f}{\partial S_n}}{\prod_{2\le i<j\le n}(x_i - x_j)}$$

Eliminating quantities: $\prod_{2\le i<j\le n}(x_i - x_j)$, result:

$$\frac{\frac{\partial f}{\partial S_1}}{x_1^{n-1}} = \cdots = \frac{\frac{\partial f}{\partial S_k}}{x_1^{n-k}} = \cdots = \frac{\frac{\partial f}{\partial S_n}}{1}$$

From here, we have:

$$\frac{\partial f}{\partial S_k} = x_1^{n-k}\lambda, \ k=\overline{1,n}, \ \lambda \in \mathbf{R}.$$

If $x_1=x_2=\ldots=x_p$, how, in this case: $S_k^1 = S_k^2 = \cdots = S_k^p$, k=$\overline{1,n}$, the system becomes:

$$\begin{cases} x_1^{n-1} + x_1^{n-2} S_1^p + \cdots + S_{n-1}^p = 0 \\ \cdots \\ x_1^{n-1} + x_1^{n-2} S_1^n + \cdots + S_{n-1}^n = 0 \end{cases}$$

Considering the equation:

$x^{n-1} + S_1^k x^{n-2} + S_2^k x^{n-3} + \cdots + S_{n-1}^k = 0$, k=$\overline{p,n}$ it becomes:

$(x + x_1) \dots (\widehat{x + x_k}) \dots (x + x_n) = 0$, k=$\overline{p,n}$.

As x_1 is the root of this equation, it follows:

$\underbrace{(2x_1) \dots (2x_1)}_{p-1\ times} (x_1 + x_{p+1}) \dots (x_1 + x_n) = 0,$

$\underbrace{(2x_1) \dots (2x_1)}_{p\ times} (x_1 + x_{p+1}) \dots (\widehat{x_1 + x_k}) \dots (x_1 + x_n) = 0$, k=$\overline{p+1,n}$.

Because $x_1 \neq 0$ follow:

$(x_1 + x_{p+1}) \dots (x_1 + x_n) = 0,$

$(x_1 + x_{p+2}) \dots (x_1 + x_n) = 0,$

...

$(x_1 + x_{p+1}) \dots (x_1 + x_{n-1}) = 0.$

From the first, if for example (after a possible renumbering) $x_{p+1} = -x_1$, from the second it results (after a possible renumbering) that $x_{p+2} = -x_1$, the rest being identically satisfied.

Therefore, the stationary points are: $x_1 = \alpha$,...,$x_p = \alpha$, $x_{p+1} = -\alpha$, $x_{p+2} = -\alpha$, $x_k \in \mathbf{R}$, k=$\overline{p+3,n}$, $\alpha \in \mathbf{R}$.

If all variables are positive then there are no variables x_{p+1} therefore: $x_1 = \cdots = x_n = \alpha$, $\alpha > 0$.

In this case: $S_k^i = \left(C_n^k - C_{n-1}^{k-1}\right)\alpha^k = C_{n-1}^k \alpha^k$, iar $S_k^{i,j} = C_{n-2}^k \alpha^k$. Thus:

$$d^2 f = \sum_{i,j=1}^n \left(\sum_{k,s=1}^n \frac{\partial^2 f}{\partial S_k \partial S_s} S_{k-1}^i S_{s-1}^j + \left(1 - \delta_{ij}\right) \sum_{k=1}^n \frac{\partial f}{\partial S_k} S_{k-2}^{i,j} \right) dx_i dx_j =$$

$$\sum_{i,j=1}^n \left(\sum_{k,s=1}^n \frac{\partial^2 f}{\partial S_k \partial S_s} C_{n-1}^{k-1} C_{n-1}^{s-1} \alpha^{k+s-2} + \left(1 - \delta_{ij}\right) \sum_{k=2}^n \frac{\partial f}{\partial S_k} C_{n-2}^{k-2} \alpha^{k-2} \right) dx_i dx_j.$$

If $f(S_1, \dots, S_n) = g\left(S_{i_1}, \dots, S_{i_p}\right)$ where $S_{i_1}, \dots, S_{i_p}$ actually appear in the expression of f then, noting M=$\{i_1, \dots, i_p\}$ we obtain (due to the identical cancellation of the partial derivatives corresponding to the variables x_k, k$\notin$M):

$$d^2 f = \sum_{i,j=1}^n \left(\sum_{\substack{k,s=1 \\ k,s \in M}}^n \frac{\partial^2 f}{\partial S_k \partial S_s} C_{n-1}^{k-1} C_{n-1}^{s-1} \alpha^{k+s-2} + \left(1 - \delta_{ij}\right) \sum_{\substack{k=2 \\ k \in M}}^n \frac{\partial f}{\partial S_k} C_{n-2}^{k-2} \alpha^{k-2} \right) dx_i dx_j.$$

In particular $\frac{\partial^2 f}{\partial S_k \partial S_s}$=0, k≠s then:

$$d^2f = \sum_{i,j=1}^{n}\left(\sum_{\substack{k=1\\ k\in M}}^{n}\frac{\partial^2 f}{\partial S_k^2}\left(C_{n-1}^{k-1}\right)^2\alpha^{2k-2} + \left(1-\delta_{ij}\right)\sum_{\substack{k=2\\ k\in M}}^{n}\frac{\partial f}{\partial S_k}C_{n-2}^{k-2}\alpha^{k-2}\right)dx_i dx_j.$$

So we can formulate the following:

Main theorem

Let be a function of fundamental symmetric polynomials: f:**R**^{n}→**R**, $f(S_1,\dots,S_n)$, $x_1,\dots,x_n>0$.

Extremely local points satisfy either condition: $\frac{\partial f}{\partial S_k}$=0, k=$\overline{1,n}$, either $x_1 = \cdots = x_n$.

The point x_0=$(x_1^0,\dots,x_n^0)$ is a local minimum/maximum if d^2f is positively/negatively defined.

Remark 1

Since the global extreme points are the extremes of the local points, the theorem determines the absolute maximum/minimum of the function.

Remark 2

Considering a symmetric polynomial function P, namely P∈**R**[$X_1,\dots,X_n$] such that ∀σ∈S_n: $P\left(X_{\sigma(1)},\dots,X_{\sigma(n)}\right)= P(X_1,\dots,X_n)$, we define the lexicographical order of its monomials by: $a_{i_1\dots i_n}X_1^{i_1}\dots X_n^{i_n} \succcurlyeq b_{j_1\dots j_n}X_1^{j_n}\dots X_n^{j_n}$ ⇔ ∃p=$\overline{1,n}$ a.î. $i_1=j_1,\dots,i_{p-1}=j_{p-1}$, $i_p\geq j_p$ and the strict lexicographical order by: $a_{i_1\dots i_n}X_1^{i_1}\dots X_n^{i_n} \succ b_{j_1\dots j_n}X_1^{j_1}\dots X_n^{j_n}$ ⇔ ∃p=$\overline{1,n}$ a.î. $i_1=j_1,\dots,i_{p-1}=j_{p-1}$, $i_p>j_p$, the longest term in the sense of the lexicographical order is called the main term. If the main term is $a_{i_1\dots i_n}X_1^{i_1}\dots X_n^{i_n}$ it is shown that $i_1\geq i_2\geq\dots\geq i_n$.

We have the following:

Theorem

Considering P∈**R**[$X_1,\dots,X_n$] a symmetric polynomial then ∃!Q∈**R**[$X_1,\dots,X_n$] such that P=Q($S_1,\dots,S_n$).

Waring's method is used to determine the polynomial Q in this way:

- we lexicographically order the polynomial P;
- considering the main term: $a_{i_1\dots i_n}X_1^{i_1}\dots X_n^{i_n}$ we construct the polynomial: R_1=$a_{i_1\dots i_n}s_1^{i_1-i_2}s_2^{i_2-i_3}\dots s_{n-1}^{i_{n-1}-i_n}s_n^{i_n}$ which is symmetrical;
- after subtraction, we obtain the polynomial P_1=P-R_1 which we order

lexicographically again and return to the previous step;

- in the end: P=R_1+...+R_k+b (*if R_k is the last polynomial thus constructed, and b is the possible free term of P*).

After all this, it follows that the main theorem is also valid for all symmetric polynomial functions, regardless of whether the fundamental symmetric polynomials appear explicitly or not.

Particular cases (for all equal variables with α):

n=2

$$d^2f = \sum_{i,j=1}^{2}\left(\frac{\partial^2 f}{\partial S_1^2} + 2\frac{\partial^2 f}{\partial S_1 \partial S_2}\alpha + \frac{\partial^2 f}{\partial S_2^2}\alpha^2\right)dx_i dx_j.$$

n=3

$$d^2f = \sum_{i,j=1}^{3}\left(\frac{\partial^2 f}{\partial S_1^2} + 4\frac{\partial^2 f}{\partial S_1 \partial S_2}\alpha + 2\frac{\partial^2 f}{\partial S_1 \partial S_3}\alpha^2 + 4\frac{\partial^2 f}{\partial S_2^2}\alpha^2 + 4\frac{\partial^2 f}{\partial S_2 \partial S_3}\alpha^3 + \frac{\partial^2 f}{\partial S_3^2}\alpha^4 + (1-\delta_{ij})\left(\frac{\partial f}{\partial S_2} + \frac{\partial f}{\partial S_3}\alpha\right)\right)dx_i dx_j.$$

3. Applications

3.1. Let's prove the known inequality: $\left(\frac{x_1+\cdots+x_n}{n}\right)^2 \le \frac{x_1^2+\cdots+x_n^2}{n}$, $x_1,\ldots,x_n>0$.

Proof

Let f:$\mathbf{R}^2\to\mathbf{R}$, $f(S_1,S_2)=\left(\frac{S_1}{n}\right)^2 - \frac{S_1^2-2S_2}{n}$.

We have $\frac{\partial f}{\partial S_1} = \frac{2(1-n)S_1}{n^2}$, $\frac{\partial f}{\partial S_2} = \frac{2}{n}$, $\frac{\partial^2 f}{\partial S_1^2}=\frac{2(1-n)}{n^2}$, $\frac{\partial^2 f}{\partial S_1 \partial S_2}=0$, $\frac{\partial^2 f}{\partial S_2^2}=0$.

How $\frac{\partial f}{\partial S_2} = \frac{2}{n}\neq 0$ it turns out that the only stationary point is (α,...,α), α∈**R**

From here: $d^2f = \sum_{i,j=1}^{n}\left(\frac{2(1-n)}{n^2} + (1-\delta_{ij})\frac{2}{n}\right)dx_i dx_j = \sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{2}{n^2}dx_i dx_j + \sum_{i=1}^{n}\frac{2(1-n)}{n^2}dx_i^2$.

The Hessian matrix is:

$H_f=\frac{1}{n^2}\begin{pmatrix} 2(1-n) & 1 & \cdots & 1 \\ 1 & 2(1-n) & \cdots & 1 \\ \ldots & \ldots & \cdots & \ldots \\ 1 & 1 & \cdots & 2(1-n) \end{pmatrix}$ whose main diagonal determinants are alternated, therefore d^2f is negatively defined. The point is the local maximum.

Therefore, how $S_1=n\alpha$, $S_2=C_n^2\alpha^2=\frac{n(n-1)}{2}\alpha^2$ we get:

$f(S_1,S_2)=\left(\frac{S_1}{n}\right)^2-\frac{S_1^2-2S_2}{n}\leq f\left(n\alpha,\frac{n(n-1)}{2}\alpha^2\right)=\alpha^2-\frac{n^2\alpha^2-2\frac{n(n-1)}{2}\alpha^2}{n}=\alpha^2-\frac{n\alpha^2}{n}=0$

Therefore, inequality is prooved.

3.2. Prove Goughens Inequality:

$(1+x_1)(1+x_2)\ldots(1+x_n)\geq\left(1+\sqrt[n]{x_1x_2\ldots x_n}\right)^n$ $\forall x_1,\ldots,x_n\geq 0$, $n\in\mathbf{N}$, $n\geq 2$.

Proof

Obviously, if one of the variables is null (for example $x_1=0$) then:

$(1+x_2)\ldots(1+x_n)\geq 1$ which is obviously true.

So, let $x_1,\ldots,x_n>0$ and the function: $f:\mathbf{R}^n\to\mathbf{R}$, $f(S_1,\ldots,S_n)=1+S_1+\cdots+S_n-\left(1+\sqrt[n]{S_n}\right)^n$

We have $\frac{\partial f}{\partial S_k}=1$, $k=\overline{1,n-1}$, $\frac{\partial f}{\partial S_n}=1-\left(1+\frac{1}{\sqrt[n]{S_n}}\right)^{n-1}$, $\frac{\partial^2 f}{\partial S_k^2}=0$, $k=\overline{1,n-1}$,

$\frac{\partial^2 f}{\partial S_n^2}=\frac{n-1}{n}\left(1+\frac{1}{\sqrt[n]{S_n}}\right)^{n-2}\frac{1}{\sqrt[n]{S_n}^{n+1}}$, $\frac{\partial^2 f}{\partial S_k\partial S_p}=0$, $k\neq p=\overline{1,n}$.

Because $\frac{\partial f}{\partial S_k}=1\neq 0$ it turns out that the only stationary point is $(\alpha,\ldots,\alpha)$, $\alpha\in\mathbf{R}$.

In this case, because $S_n=\alpha^n$ we get: $\frac{\partial f}{\partial S_k}=1$, $k=\overline{1,n-1}$, $\frac{\partial f}{\partial S_n}=1-\left(\frac{\alpha+1}{\alpha}\right)^{n-1}$,

$\frac{\partial^2 f}{\partial S_k^2}=0$, $k=\overline{1,n-1}$, $\frac{\partial^2 f}{\partial S_n^2}=\frac{n-1}{n}\left(\frac{\alpha+1}{\alpha}\right)^{n-2}\frac{1}{\alpha^{n+1}}$, $\frac{\partial^2 f}{\partial S_k\partial S_p}=0$, $k\neq p=\overline{1,n}$.

But:

$d^2f=\sum_{i,j=1}^{n}\left(\frac{\partial^2 f}{\partial S_n^2}\alpha^{2n-2}+\left(1-\delta_{ij}\right)\left(\sum_{k=2}^{n-1}C_{n-2}^{k-2}\alpha^{k-2}+\frac{\partial f}{\partial S_n}\alpha^{n-2}\right)\right)dx_idx_j=$

$\sum_{i,j=1}^{n}\left(\frac{\partial^2 f}{\partial S_n^2}\alpha^{2n-2}+\left(1-\delta_{ij}\right)\left((\alpha+1)^{n-2}+\left(\frac{\partial f}{\partial S_n}-1\right)\alpha^{n-2}\right)\right)dx_i dx_j=$

$\sum_{i,j=1}^{n}\left(\frac{n-1}{n}\left(\frac{\alpha+1}{\alpha}\right)^{n-2}\frac{1}{\alpha^{n+1}}\alpha^{2n-2}+\left(1-\delta_{ij}\right)\left((\alpha+1)^{n-2}-\right.\right.$

$\left.\left.\left(\frac{\alpha+1}{\alpha}\right)^{n-1}\alpha^{n-2}\right)\right)dx_i dx_j=$

$\sum_{i,j=1}^{n}\left(\frac{n-1}{n}(\alpha+1)^{n-2}\frac{1}{\alpha}+\left(1-\delta_{ij}\right)\left((\alpha+1)^{n-2}-\frac{1}{\alpha}(\alpha+1)^{n-1}\right)\right)dx_i dx_j=$

$\frac{(\alpha+1)^{n-2}}{\alpha}\sum_{i,j=1}^{n}\left(\frac{n-1}{n}+\left(1-\delta_{ij}\right)\left(\alpha-(\alpha+1)\right)\right)dx_i dx_j=$

$\frac{(\alpha+1)^{n-2}}{\alpha}\sum_{i,j=1}^{n}\left(-\frac{1}{n}+\delta_{ij}\right)dx_i dx_j=\frac{(\alpha+1)^{n-2}}{\alpha n}\left(\sum_{i,j=1}^{n}(n-1)dx_i^2-2\sum_{\substack{i,j=1\\ i<j}}^{n}dx_i dx_j\right)$

The Hessian matrix is:

$H_f=\frac{(\alpha+1)^{n-2}}{\alpha n}\begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \ldots & \ldots & \cdots & \ldots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$ whose main diagonal determinants are positive, therefore d^2f is positively defined. The point is the local minimum.

But $f(S_1,\ldots,S_n)=1+C_n^1\alpha+C_n^2\alpha^2\ldots+C_n^n\alpha^n-(1+\alpha)^n=0$ therefore: $f(S_1,\ldots,S_n)\geq 0$ and inequality is demonstrated.

3.3. Let's Prove the Known Inequality: $\frac{x_1+\cdots+x_n}{n}\geq\sqrt[n]{x_1\ldots x_n}$ $\forall x_1,\ldots,x_n\geq 0$, $n\in\mathbf{N}$, $n\geq 2$.

Proof

Obviously, if one of the variables is null (for example $x_1=0$) then: $\frac{x_2+\cdots+x_n}{n}\geq 0$ which is obviously true.

So, let $x_1,\ldots,x_n>0$ and the function: $f:\mathbf{R}^n\to\mathbf{R}$, $f(S_1,\ldots,S_n)=\frac{S_1}{n}-\sqrt[n]{S_n}$.

We have $\frac{\partial f}{\partial S_1}=\frac{1}{n}$, $\frac{\partial f}{\partial S_n}=-\frac{1}{n\sqrt[n]{S_n}^{n-1}}$, $\frac{\partial^2 f}{\partial S_1^2}=0$, $\frac{\partial^2 f}{\partial S_1\partial S_n}=0$, $\frac{\partial^2 f}{\partial S_n^2}=-\frac{1-n}{n^2\sqrt[n]{S_n}^{2n-1}}$ the others being null.

How $\frac{\partial f}{\partial S_1}=\frac{1}{n}\neq 0$ it turns out that the only stationary point is $(\alpha,\ldots,\alpha)$, $\alpha\in\mathbf{R}$.

In this case, because $S_n = \alpha^n$ we have:

$\frac{\partial f}{\partial S_1} = \frac{1}{n}, \frac{\partial f}{\partial S_n} = -\frac{1}{n\alpha^{n-1}}, \frac{\partial^2 f}{\partial S_1^2}$=0, $\frac{\partial^2 f}{\partial S_1 \partial S_n}$=0, $\frac{\partial^2 f}{\partial S_n^2}$=$\frac{n-1}{n^2\alpha^{2n-1}}$ and:

$d^2 f = \sum_{i,j=1}^{n}\left(\frac{n-1}{n^2\alpha^{2n-1}}\alpha^{2n-2} - (1-\delta_{ij})\frac{1}{n\alpha^{n-1}}\alpha^{n-2}\right)dx_i dx_j$=

$\frac{1}{n\alpha}\sum_{i,j=1}^{n}\left(\frac{n-1}{n} - (1-\delta_{ij})\right)dx_i dx_j$=$\frac{1}{n^2\alpha}\left(\sum_{i,j=1}^{n}(n-1)dx_i^2 - 2\sum_{\substack{i,j=1\\ i<j}}^{n} dx_i dx_j\right)$.

The quadratic form in parentheses is identical to the one in 2) so d^2f is positively defined. The point is the local minimum.

We have now: $f(S_1, \dots, S_n)$=$\frac{n\alpha}{n} - \alpha$=0 thus $f(S_1, \dots, S_n)$≥0.

3.4. Let's Prove the Known Inequality: $\frac{3(x^3+y^3+z^3)}{xyz} \geq (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ ∀x,y,z>0

Proof

Let's note $S_1 = x + y + z$, $S_2 = xy + xz + yz$, $S_3 = xyz$.

Considering the equation $t^3 - S_1t^2 + S_2t - S_3 = 0$ it has the roots x,y,z therefore:

$\sum x^3 = S_1\sum x^2 - S_2\sum x + 3S_3$=$S_1(S_1^2 - 2S_2) - S_1S_2 + 3S_3$=$S_1^3 - 3S_1S_2 + 3S_3$.

Inequality becomes:

$3(S_1^3 - 3S_1S_2 + 3S_3)$≥$S_1S_2$.

Let the function: f:$\mathbf{R}^3$→$\mathbf{R}$, $f(S_1, S_2, S_3)$=$3S_1^3 - 9S_1S_2 + 9S_3 - S_1S_2$=$3S_1^3 - 10S_1S_2 + 9S_3$.

We have:

$\frac{\partial f}{\partial S_1} = 9S_1^2 - 10S_2, \frac{\partial f}{\partial S_2} = -10S_1, \frac{\partial f}{\partial S_3} = 9, \frac{\partial^2 f}{\partial S_1^2}$=$18S_1$, $\frac{\partial^2 f}{\partial S_2^2}$=0, $\frac{\partial^2 f}{\partial S_3^2}$=0, $\frac{\partial^2 f}{\partial S_1 \partial S_2}$=-10, $\frac{\partial^2 f}{\partial S_1 \partial S_3}$=0, $\frac{\partial^2 f}{\partial S_2 \partial S_3}$=0.

How $\frac{\partial f}{\partial S_3} = 9$≠0 it turns out that the only stationary point is (α,α,α), α∈**R**.

In this case, because $S_1 = 3\alpha$, $S_2 = 3\alpha^2$, $S_3 = \alpha^3$ we get:

$\frac{\partial f}{\partial S_1} = 51\alpha^2, \frac{\partial f}{\partial S_2} = -30\alpha, \frac{\partial f}{\partial S_3} = 9, \frac{\partial^2 f}{\partial S_1^2}$=$54\alpha$, $\frac{\partial^2 f}{\partial S_2^2}$=0, $\frac{\partial^2 f}{\partial S_3^2}$=0, $\frac{\partial^2 f}{\partial S_1 \partial S_2}$=-10, $\frac{\partial^2 f}{\partial S_1 \partial S_3}$=0,

$\frac{\partial^2 f}{\partial S_2 \partial S_3}$=0 and:

$$d^2 f = \alpha\left(\sum_{i=1}^{3} 14 dx_i^2 - 2\sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{7}{2} dx_i dx_j\right).$$

The Hessian matrix is:

$H_f=\frac{\alpha}{2}\begin{pmatrix} 28 & -7 & -7 \\ -7 & 28 & -7 \\ -7 & -7 & 28 \end{pmatrix}$. But $\Delta_1=28\frac{\alpha}{2}$, $\Delta_2=735\left(\frac{\alpha}{2}\right)^2$, $\Delta_3=17150\left(\frac{\alpha}{2}\right)^3$

As the main diagonal determinants are positive, d^2f is positively defined. The point is the local minimum. We now have: $f(S_1, S_2, S_3)=81\alpha^3 - 90\alpha^3 + 9\alpha^3=0$ therefore: $f(S_1, S_2, S_3)\geq 0$.

3.5. Let's Prove the Known Inequality: $\frac{a+b}{c^2}+\frac{b+c}{a^2}+\frac{a+c}{b^2}\geq\frac{9}{a+b+c}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ ∀a,b,c>0

Proof

We can easily see that: $\frac{a+b}{c^2}+\frac{b+c}{a^2}+\frac{a+c}{b^2}=\frac{-2S_1^2S_3-S_2S_3+S_1S_2^2}{S_3^2}$ and inequality becomes:

$$\frac{-2S_1^2S_3-S_2S_3+S_1S_2^2}{S_3^2} \geq \frac{9}{S_1}+\frac{S_2}{S_3} \Leftrightarrow -2S_1^3S_3 - S_1S_2S_3 + S_1^2S_2^2 \geq 9S_3^2 + S_1S_2S_3 \Leftrightarrow$$

$$-2S_1^3S_3 - 2S_1S_2S_3 + S_1^2S_2^2 - 9S_3^2 \geq 0.$$

Let the function: f:**R**³→**R**, $f(S_1, S_2, S_3)=-2S_1^3S_3 - 2S_1S_2S_3 + S_1^2S_2^2 - 9S_3^2$.

We have:

$\frac{\partial f}{\partial S_1} = -6S_1^2S_3 - 2S_2S_3 + 2S_1S_2^2$, $\frac{\partial f}{\partial S_2} = -2S_1S_3 + 2S_1^2S_2$, $\frac{\partial f}{\partial S_3} = -2S_1^3 - 2S_1S_2 - 18S_3$,

$\frac{\partial^2 f}{\partial S_1^2}=-12S_1S_3 + 2S_2^2$, $\frac{\partial^2 f}{\partial S_2^2}=2S_1^2$, $\frac{\partial^2 f}{\partial S_3^2}=-18$, $\frac{\partial^2 f}{\partial S_1 \partial S_2}=-2S_3 + 4S_1S_2$, $\frac{\partial^2 f}{\partial S_1 \partial S_3}=-6S_1^2 - 2S_2$, $\frac{\partial^2 f}{\partial S_2 \partial S_3}=-2S_1$.

First, we will solve: $\frac{\partial f}{\partial S_k}=0$, $k=\overline{1,3}$.

$$\begin{cases} \dfrac{\partial f}{\partial S_1} = -6S_1^2S_3 - 2S_2S_3 + 2S_1S_2^2 = 0 \\ \dfrac{\partial f}{\partial S_2} = -2S_1S_3 + 2S_1^2S_2 = 0 \\ \dfrac{\partial f}{\partial S_3} = -2S_1^3 - 2S_1S_2 - 18S_3 = 0 \end{cases}$$

From the second: $S_3 = S_1S_2$ and of the first: $-6S_1^3S_2 = 0$ which give us: S_1=0 or S_2=0 – contradiction with a,b,c>0.

Therefore: a=α, b=α, c=α, α>0. But S_1=3α, S_2=3α^2, S_3=α^3 implies:

$\frac{\partial f}{\partial S_1} = -6\alpha^5, \frac{\partial f}{\partial S_2} = 48\alpha^4, \frac{\partial f}{\partial S_3} = -90\alpha^3,$

$\frac{\partial^2 f}{\partial S_1^2}$=$-18\alpha^4$, $\frac{\partial^2 f}{\partial S_2^2}$=$18\alpha^2$, $\frac{\partial^2 f}{\partial S_3^2}$=$-18$, $\frac{\partial^2 f}{\partial S_1 \partial S_2}$=$34\alpha^3$, $\frac{\partial^2 f}{\partial S_1 \partial S_3}$=$-60\alpha^2$, $\frac{\partial^2 f}{\partial S_2 \partial S_3}$=$-6\alpha$.

$d^2f = \sum_{i,j=1}^{3}\left(\frac{\partial^2 f}{\partial S_1^2} + 4\frac{\partial^2 f}{\partial S_1\partial S_2}\alpha + 2\frac{\partial^2 f}{\partial S_1\partial S_3}\alpha^2 + 4\frac{\partial^2 f}{\partial S_2^2}\alpha^2 + 4\frac{\partial^2 f}{\partial S_2\partial S_3}\alpha^3 + \frac{\partial^2 f}{\partial S_3^2}\alpha^4 + (1-\delta_{ij})\left(\frac{\partial f}{\partial S_2} + \frac{\partial f}{\partial S_3}\alpha\right)\right)dx_idx_j$=$\sum_{i,j=1}^{3}\left(-18\alpha^4 + 136\alpha^4 - 120\alpha^4 + 72\alpha^4 - 24\alpha^4 - 18\alpha^4 + (1-\delta_{ij})(48\alpha^4 - 90\alpha^4)\right)dx_idx_j$=$\sum_{i,j=1}^{3}\left(183\alpha^4 - (1-\delta_{ij})42\alpha^4\right)dx_idx_j$=

$\alpha^4\left(\sum_{i=1}^{3} 183dx_i^2 + 2\sum_{\substack{i,j=1 \\ i<j}}^{3}\frac{141}{2}dx_idx_j\right).$

The Hessian matrix is:

$H_f=\alpha^4\begin{pmatrix} 183 & -\frac{141}{2} & -\frac{141}{2} \\ -\frac{141}{2} & 183 & -\frac{141}{2} \\ -\frac{141}{2} & -\frac{141}{2} & 183 \end{pmatrix}$. But $\Delta_1=183\alpha^4$, $\Delta_2=28518.75\alpha^8$, $\Delta_3=2699014.5\alpha^{12}$.

As the main diagonal determinants are positive, d^{2}f is positively defined. The point is the local minimum. We now have: $f(S_1,S_2,S_3)$=$-54\alpha^6 - 18\alpha^6 + 81\alpha^6 - 9\alpha^6$=0 therefore: $f(S_1,S_2,S_3)\geq 0$.

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