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Determination of Conditional Extremes of the Functions of Fundamental Symmetric Polynomials

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Abstract: The paper demonstrates how the conditional extremes of the functions of fundamental symmetric polynomials, in the case of the positivity of variables, are obtained either by canceling the partial derivatives in relation to the polynomials, or by the equality of all variables.

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1. Introduction

Let $x_1,...,x_n \in \mathbf{R}$, $n \in \mathbf{N}$, $n \geq 2$ and fundamental symmetric polynomials of order k:

$$S_k=\sum_{1\le i_1<\cdots<i_k\le n} x_{i_1} \dots x_{i_k},\ k=\overline{1,n}.$$

We'll assume that $x_1,...,x_n \neq 0$, otherwise considering only non-zero variables with n corresponding to their number.

Let's also note, $S_0=1$, $S_{-p}=0$, $S_{n+p}=0$ $\forall p\geq 1$ and $S_k^{p_1,\dots,p_s}$, $k=\overline{1,n}$, $1 \le p_1 \ne \cdots \ne p_s \le n$, fundamental symmetric polynomials of order k in $x_1,\dots,\widehat{x_{p_1}},\dots,\widehat{x_{p_s}}\dots,x_n$ where ^ means that the factor is missing and, as above, $S_0^{p_1,\dots,p_s}=1$, $S_{-p}^{p_1,\dots,p_s}=0$ $\forall p\geq 1$, $S_p^{p_1,\dots,p_s}=0$ if p>n-s.

We have:

$$S_k^{p_1,\dots,p_s}=\sum_{1\le i_1<\cdots<\widehat{p_1}<\cdots<\widehat{p_s}<\cdots i_k\le n} x_{i_1}\dots\widehat{x_{p_1}}\dots\widehat{x_{p_s}}\dots x_{i_k}=x_j S_{k-1}^{j,p_1,\dots,p_s}+S_k^{j,p_1,\dots,p_s},$$

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$j \neq p_1 \neq \cdots \neq p_s$.

$S_k^{i,p_1,\dots,p_s} - S_k^{j,p_1,\dots,p_s} = x_j S_{k-1}^{j,i,p_1,\dots,p_s} + S_k^{j,i,p_1,\dots,p_s} - x_i S_{k-1}^{j,i,p_1,\dots,p_s} - S_k^{j,i,p_1,\dots,p_s} = (x_j - x_i) S_{k-1}^{j,i,p_1,\dots,p_s}$, $i \neq j \neq p_1 \neq \cdots \neq p_s$.

from where:

$\frac{\partial S_k}{\partial x_i} = \sum_{1 \le i_1 < \cdots < \hat{\imath} < \cdots < i_k \le n} x_{i_1} \dots \widehat{x_i} \dots x_{i_k} = S_{k-1}^{i}$, k=$\overline{0,n}$, i=$\overline{1,n}$,

$\frac{\partial S_k^{p_1,\dots,p_s}}{\partial x_i} = S_{k-1}^{i,p_1,\dots,p_s}$.

2. Main Theorem

Let $x_1,\dots,x_n \in \mathbf{R}$, $n \in \mathbf{N}$, $n \ge 2$. Let be a function of fundamental symmetric polynomials: $f:\mathbf{R}^n \to \mathbf{R}$, $f(S_1,\dots,S_n)$ and a restriction of variables $g(S_1,\dots,S_n) = 0$.

Let the Lagrangeian of the functions: $\Phi = f - \lambda g$ where Φ is a function of $\lambda, x_1,\dots,x_n$. If a point is a local extreme with connections then the relationships take place: $\frac{\partial \Phi}{\partial x_i} = 0$, i=$\overline{1,n}$, $\frac{\partial \Phi}{\partial \lambda} = 0$ that is: $\frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} = 0$, i=$\overline{1,n}$, g=0.

Let also the null space of dg in a stationary point $x^0 = (x_1^0,\dots,x_n^0)$ of Φ that is the Null=$\left\{x \in \boldsymbol{R}^n \,\middle|\, \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x^0) x_i = 0\right\}$.

Because $d^2\Phi = \frac{\partial^2 \Phi}{\partial \lambda^2} d\lambda^2 + \sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial \lambda \partial x_i} d\lambda dx_i + \sum_{i,j=1}^{n} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} dx_i dx_j =$

$-\sum_{i=1}^{n} \frac{\partial g}{\partial x_i} d\lambda dx_i + \sum_{i,j=1}^{n} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} dx_i dx_j$ we have the following:

Theorem ([3])

Considering a stationary point (λ^0, x^0) where $x^0 = (x_1^0,\dots,x_n^0)$ of Φ we have:

a) If f has a local maximum with constraint g=β, β∈**R** then $d^2\Phi(x^0)(x,x) \le 0$ ∀x∈Null;

b) If f has a local minimum with constraint g=β, β∈**R** then $d^2\Phi(x^0)(x,x) \ge 0$ ∀x∈Null;

c) If $d^2\Phi(x^0)(x,x) < 0$ ∀x∈Null-{0} then f has a strict local maximum with constraint g=β;

d) If $d^2\Phi(x^0)(x,x) > 0$ ∀x∈Null-{0} then f has a strict local minimum with constraint g=β;

e) If $d^2\Phi(x^0)(x,x)$ has both signs for $x_1,x_2\in$Null-{0} then f hasn't a strict local extreme with constraint g=β.

Considering the bordered Hessian matrix computed in (λ^0, x^0):

$$H=\begin{pmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \frac{\partial \Phi}{\partial x_1^2} & \frac{\partial \Phi}{\partial x_1 \partial x_2} & \cdots & \frac{\partial \Phi}{\partial x_1 \partial x_n} \\ \frac{\partial g}{\partial x_2} & \frac{\partial \Phi}{\partial x_2 \partial x_1} & \frac{\partial \Phi}{\partial x_2^2} & \cdots & \frac{\partial \Phi}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g}{\partial x_n} & \frac{\partial \Phi}{\partial x_n \partial x_1} & \frac{\partial \Phi}{\partial x_n \partial x_2} & & \frac{\partial \Phi}{\partial x_n^2} \end{pmatrix}$$

if we note Δ_k the main diagonal minors (of order k), then:

- if Δ_i, i=$\overline{3, n+1}$ have alternating signs, first Δ_3 having the sign + then the point is of local maximum;
- if Δ_i, i=$\overline{3, n+1}$ have the same sign "-" then the point is the local minimum.

Therefore, when we investigate the nature of a stationary point, we will construct the bordered Hessian matrix. If one of the minors Δ_i is zero, then we will use the construction of $d^2\Phi$ and the theorem above.

We have now: $\frac{\partial \Phi}{\partial x_i}=\sum_{k=1}^{n} \frac{\partial \Phi}{\partial S_k}\frac{\partial S_k}{\partial x_i}=\sum_{k=1}^{n} \frac{\partial \Phi}{\partial S_k} S_{k-1}^i$,

$\frac{\partial g}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial g}{\partial S_k}\frac{\partial S_k}{\partial x_i}=\sum_{k=1}^{n} \frac{\partial g}{\partial S_k} S_{k-1}^i$ and also:

$\frac{\partial^2 \Phi}{\partial x_i \partial x_j}=\frac{\partial}{\partial x_j}\sum_{k=1}^{n} \frac{\partial \Phi}{\partial S_k} S_{k-1}^i=\sum_{k,s=1}^{n} \frac{\partial^2 \Phi}{\partial S_k \partial S_s} S_{k-1}^i S_{s-1}^j + \sum_{k=1}^{n} \frac{\partial \Phi}{\partial S_k} S_{k-2}^{i,j}$ if i≠j and:

$\frac{\partial^2 \Phi}{\partial x_i^2}=\frac{\partial}{\partial x_i}\sum_{k=1}^{n} \frac{\partial \Phi}{\partial S_k} S_{k-1}^i=\sum_{k,s=1}^{n} \frac{\partial^2 \Phi}{\partial S_k \partial S_s} S_{k-1}^i S_{s-1}^i$.

We aim to determine the local extremes of the function f.

To determine the stationary points, Lagrange conditions becomes:

$\frac{\partial \Phi}{\partial x_i}$=0, i=$\overline{1, n}$, g=0 or, deployed:

$$\begin{cases} \frac{\partial \Phi}{\partial S_1} + \frac{\partial \Phi}{\partial S_2} S_1^1 + \cdots + \frac{\partial \Phi}{\partial S_n} S_{n-1}^1 = 0 \\ \cdots \\ \frac{\partial \Phi}{\partial S_1} + \frac{\partial \Phi}{\partial S_2} S_1^n + \cdots + \frac{\partial \Phi}{\partial S_n} S_{n-1}^n = 0 \\ g = 0 \end{cases}.$$

If all partial derivatives $\frac{\partial \Phi}{\partial S_k}$=0 at one point $(x_1^0, \dots, x_n^0)$then the first n equations are checked, therefore the problem reduce to:

$$\begin{cases} \frac{\partial \Phi}{\partial S_1} = 0 \\ \dots \\ \frac{\partial \Phi}{\partial S_n} = 0 \\ g = 0 \end{cases}.$$

Solving the system and using the bordered Hessian matrix (or $d^2\Phi$) we can find if the point is local minimum or local maximum.

If at least one of the partial derivatives $\frac{\partial \Phi}{\partial S_k}$≠0 then the system does not support only the null solution, therefore the determinant of the first n equations is null:

$$\begin{vmatrix} 1 & S_1^1 \dots & S_{n-1}^1 \\ 1 & S_1^2 \dots & S_{n-1}^2 \\ \dots & \dots \dots & \dots \\ 1 & S_1^n \dots & S_{n-1}^n \end{vmatrix} = 0.$$

Subtracting the first line from the others, results in:

$$0 = \begin{vmatrix} 1 & S_1^1 \dots & S_{n-1}^1 \\ 1 & S_1^2 \dots & S_{n-1}^2 \\ \dots & \dots \dots & \dots \\ 1 & S_1^n \dots & S_{n-1}^n \end{vmatrix} = \begin{vmatrix} 1 & S_1^1 & \dots & S_{n-1}^1 \\ 0 & (x_1 - x_2)S_0^{1,2} & \dots & (x_1 - x_2)S_{n-2}^{1,2} \\ \dots & \dots & \dots & \dots \\ 0 & (x_1 - x_n)S_0^{1,n} & \dots & (x_1 - x_n)S_{n-2}^{1,n} \end{vmatrix} =$$

$$\prod_{j=2}^{n}(x_1 - x_j) \begin{vmatrix} 1 & S_1^{1,2} \dots & S_{n-2}^{1,2} \\ 1 & S_1^{1,3} \dots & S_{n-2}^{1,3} \\ \dots & \dots \dots & \dots \\ 1 & S_1^{1,n} \dots & S_{n-2}^{1,n} \end{vmatrix} = \dots = \prod_{1 \le i < j \le n}(x_i - x_j).$$

It follows from this expression that at least two variables must be equal.

Suppose, after a possible renumbering, that $x_1=x_2$.

How, in this case: $S_k^1 = S_k^2$, k=$\overline{1,n}$the second equation of the characteristic system is identical to the first, so we have:

$$\begin{cases} \frac{\partial \Phi}{\partial S_1} + \frac{\partial \Phi}{\partial S_2} S_1^2 + \dots + \frac{\partial \Phi}{\partial S_n} S_{n-1}^2 = 0 \\ \dots \\ \frac{\partial \Phi}{\partial S_1} + \frac{\partial \Phi}{\partial S_2} S_1^n + \dots + \frac{\partial \Phi}{\partial S_n} S_{n-1}^n = 0 \\ g = 0 \end{cases}$$

In this case:

$$\frac{\frac{\partial\Phi}{\partial S_1}}{\begin{vmatrix} S_1^2 S_2^2 \cdots S_{n-2}^2 S_{n-1}^2 \\ \dots \dots \cdots \dots \dots \\ S_1^n S_2^n \cdots S_{n-2}^n S_{n-1}^n \end{vmatrix}} = \frac{\frac{\partial\Phi}{\partial S_2}}{\begin{vmatrix} S_2^2 S_3^2 \cdots S_{n-1}^2\, 1 \\ \dots \dots \cdots \dots \dots \\ S_2^n S_3^n \cdots S_{n-1}^n\, 1 \end{vmatrix}} = \frac{\frac{\partial\Phi}{\partial S_3}}{\begin{vmatrix} S_3^2 S_4^2 \cdots 1\, S_1^2 \\ \dots \dots \cdots \dots \dots \\ S_3^n S_4^n \cdots 1\, S_1^n \end{vmatrix}} = \cdots$$

$$= \frac{\frac{\partial\Phi}{\partial S_n}}{\begin{vmatrix} 1\, S_1^2 \cdots S_{n-3}^2 S_{n-2}^2 \\ \dots \dots \cdots \dots \dots \\ 1\, S_1^n \cdots S_{n-3}^n S_{n-2}^n \end{vmatrix}}$$

which is equivalent to:

$$\frac{\frac{\partial\Phi}{\partial S_1}}{x_1^{n-1}\prod_{2\le i<j\le n}(x_i - x_j)} = \cdots = \frac{\frac{\partial\Phi}{\partial S_k}}{x_1^{n-k}\prod_{2\le i<j\le n}(x_i - x_j)} = \cdots$$

$$= \frac{\frac{\partial\Phi}{\partial S_n}}{\prod_{2\le i<j\le n}(x_i - x_j)}$$

Eliminating quantities: $\prod_{2\le i<j\le n}(x_i - x_j)$, result:

$$\frac{\frac{\partial\Phi}{\partial S_1}}{x_1^{n-1}} = \cdots = \frac{\frac{\partial\Phi}{\partial S_k}}{x_1^{n-k}} = \cdots = \frac{\frac{\partial\Phi}{\partial S_n}}{1}$$

From here, we have:

$$\frac{\partial\Phi}{\partial S_k} = x_1^{n-k}\mu,\ k=\overline{1,n},\ \mu\in\mathbf{R}.$$

If $x_1=x_2=...=x_p$, how, in this case: $S_k^1 = S_k^2 = \cdots = S_k^p$, k=$\overline{1,n}$, the system becomes:

$$\begin{cases} x_1^{n-1} + x_1^{n-2}S_1^p + \cdots + S_{n-1}^p = 0 \\ \dots \\ x_1^{n-1} + x_1^{n-2}S_1^n + \cdots + S_{n-1}^n = 0 \\ g = 0 \end{cases}$$

Considering the equation:

$x^{n-1} + S_1^k x^{n-2} + S_2^k x^{n-3} + \cdots + S_{n-1}^k = 0$, k=$\overline{p,n}$it becomes:

$(x + x_1) \dots \widehat{(x + x_k)} \dots (x + x_n) = 0$, k=$\overline{p,n}$.

As x_1 is the root of this equation, it follows:

$\underbrace{(2x_1) \dots (2x_1)}_{p-1\ times}(x_1 + x_{p+1}) \dots (x_1 + x_n) = 0$,

$\underbrace{(2x_1)\ldots(2x_1)}_{p\ times}\left(x_1+x_{p+1}\right)\ldots(\widehat{x_1+x_k})\ldots(x_1+x_n)=0$, k=$\overline{p+1,n}$.

Because $x_1 \neq 0$ follow:

$\left(x_1+x_{p+1}\right)\ldots(x_1+x_n)=0,$

$\left(x_1+x_{p+2}\right)\ldots(x_1+x_n)=0,$

...

$\left(x_1+x_{p+1}\right)\ldots(x_1+x_{n-1})=0.$

From the first, if for example (after a possible renumbering) $x_{p+1}=-x_1$, from the second it results (after a possible renumbering) that $x_{p+2}=-x_1$, the rest being identically satisfied.

Therefore, the stationary points are: $x_1=\alpha$,...,$x_p=\alpha$, $x_{p+1}=-\alpha$, $x_{p+2}=-\alpha$, $x_k\in\mathbf{R}$, k=$\overline{p+3,n}$, $\alpha\in\mathbf{R}$.

If all variables are positive then there are no variables x_{p+1} therefore: $x_1=\cdots=x_n=\alpha$, $\alpha>0$.

Replacing these equal values in g=0 we will find their exact value.

In this case: $S_k^i=\left(C_n^k-C_{n-1}^{k-1}\right)\alpha^k=C_{n-1}^k\alpha^k$, iar $S_k^{i,j}=C_{n-2}^k\alpha^k$. Thus:

$\frac{\partial g}{\partial x_i}=\sum_{k=1}^n\frac{\partial g}{\partial S_k}C_{n-1}^{k-1}\alpha^{k-1},$

$\frac{\partial^2\Phi}{\partial x_i\partial x_j}=\sum_{k,s=1}^n\frac{\partial^2\Phi}{\partial S_k\partial S_s}C_{n-1}^{k-1}C_{n-1}^{s-1}\alpha^{k+s-2}+\sum_{k=2}^n\frac{\partial\Phi}{\partial S_k}C_{n-2}^{k-2}\alpha^{k-2}=$

$\sum_{k,s=1}^n\frac{\partial^2 f}{\partial S_k\partial S_s}C_{n-1}^{k-1}C_{n-1}^{s-1}\alpha^{k+s-2}+\sum_{k=2}^n\frac{\partial f}{\partial S_k}C_{n-2}^{k-2}\alpha^{k-2}-$
$\lambda\left(\sum_{k,s=1}^n\frac{\partial^2 g}{\partial S_k\partial S_s}C_{n-1}^{k-1}C_{n-1}^{s-1}\alpha^{k+s-2}+\sum_{k=2}^n\frac{\partial g}{\partial S_k}C_{n-2}^{k-2}\alpha^{k-2}\right)$ if $i\neq j$ and:

$\frac{\partial^2\Phi}{\partial x_i^2}=\sum_{k,s=1}^n\frac{\partial^2\Phi}{\partial S_k\partial S_s}C_{n-1}^{k-1}C_{n-1}^{s-1}\alpha^{k+s-2}=$

$\sum_{k,s=1}^n\frac{\partial^2 f}{\partial S_k\partial S_s}C_{n-1}^{k-1}C_{n-1}^{s-1}\alpha^{k+s-2}-\lambda\sum_{k,s=1}^n\frac{\partial^2 g}{\partial S_k\partial S_s}C_{n-1}^{k-1}C_{n-1}^{s-1}\alpha^{k+s-2}.$

By entering these expressions in the bordered Hessian matrix we will find the nature of stationary points.

Main theorem

Let be a function of fundamental symmetric polynomials: f:$\mathbf{R}^n \to \mathbf{R}$, $f(S_1, \dots, S_n)$ and a restriction of variables $g(S_1, \dots, S_n) = 0$.

Extremely local points satisfy either condition: $\begin{cases} \frac{\partial \Phi}{\partial S_1} = 0 \\ \dots \\ \frac{\partial \Phi}{\partial S_n} = 0 \\ g = 0 \end{cases}$, either $x_1 = \dots = x_n$ saitsfying $g(S_1, \dots, S_n) = 0$.

The point x_0=$(x_1^0, \dots, x_n^0)$ is a local minimum/maximum if d^2f is positively/negatively defined.

Remark 1

Since the global extreme points are the extremes of the local points, the theorem determines the absolute maximum/minimum of the function.

Remark 2

Considering a symmetric polynomial function P, namely P∈$\mathbf{R}[X_1,\dots,X_n]$ such that $\forall \sigma \in S_n$: $P(X_{\sigma(1)}, \dots, X_{\sigma(n)})$= $P(X_1, \dots, X_n)$, we define the lexicographical order of its monomials by: $a_{i_1 \dots i_n} X_1^{i_1} \dots X_n^{i_n} \succcurlyeq b_{j_1 \dots j_n} X_1^{j_1} \dots X_n^{j_n} \Leftrightarrow \exists p=\overline{1,n}$ a.î. $i_1=j_1,\dots,i_{p-1}=j_{p-1}$, $i_p \geq j_p$and the strict lexicographical order by: $a_{i_1 \dots i_n} X_1^{i_1} \dots X_n^{i_n} \succ b_{j_1 \dots j_n} X_1^{j_1} \dots X_n^{j_n} \Leftrightarrow \exists p=\overline{1,n}$ a.î. $i_1=j_1,\dots,i_{p-1}=j_{p-1}$, $i_p>j_p$, the longest term in the sense of the lexicographical order is called the main term. If the main term is$a_{i_1 \dots i_n} X_1^{i_1} \dots X_n^{i_n}$it is shown that $i_1 \geq i_2 \geq \dots \geq i_n$.

We have the following:

Theorem

Considering P∈$\mathbf{R}[X_1,\dots,X_n]$ a symmetric polynomial then ∃!Q∈$\mathbf{R}[X_1,\dots,X_n]$ such that P=Q($S_1,\dots,S_n$).

Waring's method is used to determine the polynomial Q in this way:

- we lexicographically order the polynomial P;
- considering the main term: $a_{i_1 \dots i_n} X_1^{i_1} \dots X_n^{i_n}$we construct the polynomial: R_1=$a_{i_1 \dots i_n} s_1^{i_1-i_2} s_2^{i_2-i_3} \dots s_{n-1}^{i_{n-1}-i_n} s_n^{i_n}$which is symmetrical;
- after subtraction, we obtain the polynomial P_1=P-R_1which we order lexicographically again and return to the previous step;

- in the end: P=R_1+...+R_k+b (*if R_k is the last polynomial thus constructed, and b is the possible free term of P*).

After all this, it follows that the main theorem is also valid for all symmetric polynomial functions, regardless of whether the fundamental symmetric polynomials appear explicitly or not.

3. Applications

3.1. Let's prove the known inequality: $3(x+y+z) \geq 4\left(\frac{1}{x+y}+\frac{1}{x+z}+\frac{1}{y+z}\right)^2$ ∀x,y,z>0, xyz=1.

Proof

The inequality can be written as:

$$\frac{-4S_1^4-8S_1^2S_2-4S_2^2+3S_1^3S_2^2-6S_1^2S_2S_3+3S_1S_3^2}{(S_3-S_1S_2)^2} \geq 0$$

which is equivalent with:

$$-4S_1^4-8S_1^2S_2-4S_2^2+3S_1^3S_2^2-6S_1^2S_2S_3+3S_1S_3^2 \geq 0$$

Let f:$\mathbf{R}^3$→**R**, $f(S_1,S_2,S_3)$=$-4S_1^4-8S_1^2S_2-4S_2^2+3S_1^3S_2^2-6S_1^2S_2S_3+3S_1S_3^2$ and g:$\mathbf{R}^3$→**R**, $g(S_1,S_2,S_3)$=S_3-1.

We have: *Φ*=f-λg=$-4S_1^4-8S_1^2S_2-4S_2^2+3S_1^3S_2^2-6S_1^2S_2S_3+3S_1S_3^2-\lambda(S_3-1)$

$\frac{\partial\Phi}{\partial S_1}=-16S_1^3-16S_1S_2+9S_1^2S_2^2-12S_1S_2S_3+3S_3^2$,

$\frac{\partial\Phi}{\partial S_2}=-8S_1^2-8S_2+6S_1^3S_2-6S_1^2S_3$,

$\frac{\partial\Phi}{\partial S_3}=-6S_1^2S_2+6S_1S_3-\lambda$,

$\frac{\partial^2\Phi}{\partial S_1^2}=-48S_1^2-16S_2+18S_1S_2^2-12S_2S_3$,

$\frac{\partial^2\Phi}{\partial S_2^2}=-8+6S_1^3$,

$\frac{\partial^2\Phi}{\partial S_3^2}=6S_1$,

$\frac{\partial^2\Phi}{\partial S_1\partial S_2}=-16S_1+18S_1^2S_2-12S_1S_3$,

$\frac{\partial^2\Phi}{\partial S_1 \partial S_3} = -12S_1S_2 + 6S_3$,

$\frac{\partial^2\Phi}{\partial S_2 \partial S_3} = -6S_1^2$.

The condition: $\begin{cases} \frac{\partial\Phi}{\partial S_1} = 0 \\ \frac{\partial\Phi}{\partial S_2} = 0 \\ \frac{\partial\Phi}{\partial S_3} = 0 \\ g = 0 \end{cases}$ becomes:

$$\begin{cases} -16S_1^3 - 16S_1S_2 + 9S_1^2S_2^2 - 12S_1S_2S_3 + 3S_3^2 = 0 \\ -8S_1^2 - 8S_2 + 6S_1^3S_2 - 6S_1^2S_3 = 0 \\ -6S_1^2S_2 + 6S_1S_3 - \lambda = 0 \\ S_3 - 1 = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} -16S_1^3 - 28S_1S_2 + 9S_1^2S_2^2 + 3 = 0 \\ -7S_1^2 - 4S_2 + 3S_1^3S_2 = 0 \\ -6S_1^2S_2 + 6S_1 - \lambda = 0 \\ S_3 = 1 \end{cases}$$

But $S_2 = \frac{7S_1^2}{3S_1^3 - 4}$>0 implies that $S_1 > \sqrt[3]{\frac{4}{3}}$ and: $-6S_1^9 + 11S_1^6 + 19S_1^3 + 2 = 0$ which gives us: $S_1 = 1.434$, S_2 =2.97.

In this case, the equation $t^3 - S_1t^2 + S_2t - S_3 = 0$ gives solutions not real.

The only stationary point is therefore (α,α,α), α∈**R**. Because g=$S_3 - 1$=$\alpha^3 - 1$=0 we have α=1.

In this case, because $S_1 = 3$, S_2=3, S_3=1 we have:

$\frac{\partial\Phi}{\partial S_1} = 48, \frac{\partial\Phi}{\partial S_2} = 336, \frac{\partial\Phi}{\partial S_3} = -144 - \lambda, \frac{\partial^2\Phi}{\partial S_1^2} = 66, \frac{\partial^2\Phi}{\partial S_2^2} = 154, \frac{\partial^2\Phi}{\partial S_3^2} = 18, \frac{\partial^2\Phi}{\partial S_1 \partial S_2} = 402$,

$\frac{\partial^2\Phi}{\partial S_1 \partial S_3} = -102, \frac{\partial^2\Phi}{\partial S_2 \partial S_3} = -54, \frac{\partial g}{\partial S_1}$ =0, $\frac{\partial g}{\partial S_2} = 0, \frac{\partial g}{\partial S_3} = 1$.

From here:

$\frac{\partial g}{\partial x_i}$=$S_2^i$=1 and also:

$\frac{\partial \Phi}{\partial x_i}$=576-λ, $\frac{\partial^2 \Phi}{\partial x_i \partial x_j}$=$2080 - \lambda$, $\frac{\partial^2 \Phi}{\partial x_i^2}$=1888

From characteristic system: : $\frac{\partial \Phi}{\partial x_i}$=0, i=$\overline{1,3}$, g=0 we find that λ=576 and: $\frac{\partial g}{\partial x_i}$=1, $\frac{\partial^2 \Phi}{\partial x_i \partial x_j}$=1504, $\frac{\partial^2 \Phi}{\partial x_i^2}$=1888.

The bordered Hessian matrix is:

$$H=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1888 & 1504 & 1504 \\ 1 & 1504 & 1888 & 1504 \\ 1 & 1504 & 1504 & 1888 \end{pmatrix}$$

But Δ_3=-768, Δ_4=-442368 implies that the point is local minimum. In this case:

$f(3,3,1)$=0 therefore $f(S_1,S_2,S_3)$≥0 and the inequality is prooved.

3.2. Let's prove the known inequality: $3(x^2+y^2+z^2)+x+y+z \geq 6+xy+xz+yz$∀x,y,z>0, xyz=1.

Proof

The inequality can be written as:

$$2(S_1^2-2S_2)+S_1 \geq 6+S_2$$

or equivalent:

$$2S_1^2-5S_2+S_1-6 \geq 0$$

with the condition $S_3=1$.

Let f:$\mathbf{R}^3$→**R**, $f(S_1,S_2,S_3)$=$2S_1^2-5S_2+S_1-6$ and g:$\mathbf{R}^3$→**R**, $g(S_1,S_2,S_3)$=S_3-1.

We have: *Φ*=f-λg=$2S_1^2-5S_2+S_1-6-\lambda(S_3-1)$

$\frac{\partial \Phi}{\partial S_1}=4S_1+1$, $\frac{\partial \Phi}{\partial S_2}=-5$, $\frac{\partial \Phi}{\partial S_3}=-\lambda$, $\frac{\partial^2 \Phi}{\partial S_1^2}=4$, $\frac{\partial^2 \Phi}{\partial S_2^2}=0$, $\frac{\partial^2 \Phi}{\partial S_3^2}=0$,

$\frac{\partial^2 \Phi}{\partial S_1 \partial S_2}=0$, $\frac{\partial^2 \Phi}{\partial S_1 \partial S_3}=0$, $\frac{\partial^2 \Phi}{\partial S_2 \partial S_3}=0$.

Because $\frac{\partial \Phi}{\partial S_2}=-5$, the condition: $\begin{cases} \frac{\partial \Phi}{\partial S_1}=0 \\ \frac{\partial \Phi}{\partial S_2}=0 \\ \frac{\partial \Phi}{\partial S_3}=0 \\ g=0 \end{cases}$ has no solution.

The only stationary point is therefore (α,α,α), α∈**R**. Because g=S_3-1=α^3-1=0 we have α=1.

In this case, because $S_1 = 3$, S_2=3, S_3=1 we have:

$\frac{\partial\Phi}{\partial S_1} = 13, \frac{\partial\Phi}{\partial S_2} = -5, \frac{\partial\Phi}{\partial S_3} = -\lambda, \frac{\partial^2\Phi}{\partial S_1^2} = 4, \frac{\partial^2\Phi}{\partial S_2^2} = 0, \frac{\partial^2\Phi}{\partial S_3^2} = 0, \frac{\partial^2\Phi}{\partial S_1\partial S_2} = 0, \frac{\partial^2\Phi}{\partial S_1\partial S_3} = 0, \frac{\partial^2\Phi}{\partial S_2\partial S_3} = 0,$

$\frac{\partial g}{\partial S_1}$ =0, $\frac{\partial g}{\partial S_2} = 0, \frac{\partial g}{\partial S_3} = 1.$

From here:

$\frac{\partial g}{\partial x_i}$=$S_2^i$=1 and also: $\frac{\partial \Phi}{\partial x_i}$=3-λ, $\frac{\partial^2 \Phi}{\partial x_i\partial x_j}$=4-λ, $\frac{\partial^2 \Phi}{\partial x_i^2}$=4.

From characteristic system: : $\frac{\partial \Phi}{\partial x_i}$=0, i=$\overline{1,3}$, g=0 we find that λ=3 and: $\frac{\partial g}{\partial x_i}$=1, $\frac{\partial^2 \Phi}{\partial x_i\partial x_j}$=1, $\frac{\partial^2 \Phi}{\partial x_i^2}$=4.

The bordered Hessian matrix is:

$$H=\begin{pmatrix} 0111 \\ 1411 \\ 1141 \\ 1114 \end{pmatrix}$$

But Δ_3=-6, Δ_4=-27 implies that the point is local minimum. In this case:

$f(3,3,1)$=0 therefore $f(S_1, S_2, S_3)$≥0 and the inequality is prooved.

References

Chirciu, M. (2014). *Algebraic inequalities from initiation to performance*. Pitesti: Parallel Publishing House 45.

Ioan, C. A. (2021). *Mathematics.01.Algebra*. Galati: Zigotto Publishers.

Spring D. (1985). *On the second derivative test for constrained local extrema*. Pennsylvania: American Mathematical Monthly, vol. 92, pp. 631-643.