Acta Universitatis Danubius
ŒCONOMICA

# The General Theory of Short-Term and Long-Term Costs

**Gina Ioan**[1]

**Abstract:** The theory of costs is of particular importance in the theory of the producer, giving essential information on the level of production that must be achieved within the limits of acceptable costs in the market. The axiomatic approach to this theory highlights the essential aspects that are rigorously demonstrated and that provide insight into the behavior of different types of costs.

**Keywords:** cost; short-term; long-term

**JEL Classification**:

## 1. Introduction

In a previous paper ([1]) we presented the main aspects of the cost function for a firm. We will briefly summarize the main results.

Defining on $\mathbf{R}^n$ the production space for n fixed resources as $SP=\{(x_1,...,x_n) \mid x_i \geq 0, i=\overline{1,n}\}$ where $x \in SP$, $x=(x_1,...,x_n)$ represents an ordered set of resources, we will restrict the production space to a subset $DP \subset SP$ named domain of production.

An application $Q: DP \to \mathbf{R}_+$, $(x_1,...,x_n) \to Q(x_1,...,x_n) \in \mathbf{R}_+$ $\forall (x_1,...,x_n) \in DP$ is called a production function if it satisfies the following axioms:

- The domain of production is convex;
- If all resources are zero then production is null;
- The production function admits partial derivatives of order 2, and they are continuous (the function is of class $C^2$ on DP);

[1] Dunarea de Jos University of Galati, Romania, Address: Domneasca Street 47, Galati, Romania, Corresponding author: gina.ioan@ugal.ro.

- The production function is monotonically increasing in each variable;
- The production function is quasi-concave.

Considering the production domain DP, a production function Q:DP→$\mathbf{R}_+$, $(x_1,...,x_n)\to Q(x_1,...,x_n)$ and $p_1,...,p_n$ – their prices, we call production cost relative to the consumption $x_1,...,x_n$ of factors, the quantity: $C(x_1,...,x_n)=\sum_{i=1}^{n} p_i x_i$.

For a fixed production $Q_0$, we will call the production cost relative to the production function Q for obtaining a result $Q_0$, the solution of the nonlinear programming problem:

$$\begin{cases} min(p_1x_1+\ldots+p_nx_n) \\ Q(x_1,\ldots,x_n) \geq Q_0 \\ x_1,\ldots,x_n \geq 0 \end{cases}$$

The Karush-Kuhn-Tucker conditions for the above problem, become:

$$\begin{cases} \dfrac{\frac{\partial Q}{\partial x_1}(\bar{x}_1,\ldots,\bar{x}_n)}{p_1} =\ldots= \dfrac{\frac{\partial Q}{\partial x_n}(\bar{x}_1,\ldots,\bar{x}_n)}{p_n} \\ Q(\bar{x}_1,\ldots,\bar{x}_n) = Q_0 \end{cases}$$

where the solutions of this system represent the optimal combination of production factors that ultimately determine the minimum cost: $C(p,Q_0)=p_1\bar{x}_1+\ldots+p_n\bar{x}_n$ where $p=(p_1,...,p_n)\in R_+^{*n}$.

Also, there are a lot of theorems which acceptably characterizes the production costs:

- ***Shephard Lemma*** - Considering the cost function: $C(p,Q_0)=p_1\bar{x}_1+\ldots+p_n\bar{x}_n$, of class $C^1$, the following equality occurs: $\bar{x}_k=\frac{\partial C(p,Q_0)}{\partial p_k}$, k=$\overline{1,n}$.

- ***The monotonicity property of the relative cost function at the production level*** - For a constant level of factor prices, the cost function is increasing in relation to the level of production.

- ***The monotonicity property of the cost function relative to price*** - For a fixed value of output, the cost function is increasing relative to the set of factor prices.

- ***The homogeneity property of the cost function relative to price*** - For a fixed value of output, the cost function is homogeneous of the first degree with respect to the set of factor prices.

- ***The concavity property of the cost function relative to price*** - For a fixed value of output, the cost function is concave with respect to the set of factor prices**.**

• ***Convexity property of the cost function relative to production -*** For a fixed value of prices, the cost function is convex with respect to output.

## 2. Main Categories of Costs

In the theory of costs, a number of factors appear that compete to define their different types and specificities.

If in relation to the consumption of production factors and in relation to their cost we addressed in article [1] the cost of production (CP) or the explicit cost, an indirect analysis generates the so-called **opportunity cost** (CO) or the implicit cost. This represents the maximum benefit that is given up by investing monetary resources in production. Thus, an amount of money spent on the purchase of high-performance machines could be invested in another business or even deposited in a bank with some interest. The maximum benefit brought by these alternatives will represent the opportunity cost of purchasing the new machines.

The **accounting cost** (CC) represents all the expenses incurred for supply, labor remuneration, the actual production process and other activities necessary for the proper performance of the activity.

The **economic cost** (CE) differs from the accounting one in that it adds to the latter the opportunity cost CO. We can therefore write: CE=CC+CO.

From another point of view, costs are of two types: fixed and variable.

**Fixed costs** (CF) are those costs that are independent of the value of production (rents, lighting costs, heating costs, interest, etc.) and are paid regardless of whether there is production or not.

**Quasi-fixed costs** (CCF) are those costs that are also independent of the value of production, but are paid only if there is production (for example, advertising expenses).

**Variable costs** (CV) represent the totality of production expenses that vary with production, in the same sense (expenses with raw materials and materials, wages, expenses with energy consumed during the production process, etc.).

**The total cost** (CT) of production is the sum of fixed and variable costs: CT=CF+CV.

Considering now the production domain DP, and a production function $Q:DP\to\mathbf{R}_+$, $(x_1,...,x_n)\to Q(x_1,...,x_n)$, $p_1,...,p_n$ – the prices of production factors $x_1,...,x_n$, the production cost is:

$$CT=C(x_1,...,x_n)=\sum_{i=1}^{n} p_i x_i$$

Considering a level Q of production, we call the **average fixed cost** (CFM) the value:

$$\text{CFM}=\frac{CF}{Q}$$

and represents the fixed cost per unit of product.

Now considering a given output Q, we will call the **average variable cost** (CVM) the value:

$$\text{CVM}=\frac{CV}{Q}$$

which represents the variable cost for a unit of product.

We now define the average total cost (CTM) by the formula:

$$\text{CTM}=\frac{CT}{Q}$$

representing the total cost for a single unit of product.

Because: CT=CF+CV, dividing by Q, we obtain:

$$\text{CTM}=\frac{CT}{Q}=\frac{CF}{Q}+\frac{CV}{Q}=\text{CFM}+\text{CVM}$$

It is well known that, often for the evolution of a phenomenon, its absolute expression is less edifying than its speed. Thus:

For a level Q of production, is called marginal cost (Cm) the value:

$$\text{Cm}=\frac{\partial CT}{\partial Q}$$

and represents the rate of change of total cost relative to a given output.

Let us note, in the following: $\eta_{x_i}$ - the marginal productivity relative to the production factor $x_i$, $w_{x_i}$ - the average physical productivity of each factor of production $x_i$ and $\gamma_{x_i}=\frac{\partial x_1}{\partial Q}$ - marginal coefficients of production factors $x_i$.

***Theorem***

At a given total cost, the maximum level of output is achieved if and only if:

$$\frac{\eta_{x_1}}{p_1}=\ldots=\frac{\eta_{x_n}}{p_n}$$

***Proof***

Let $Q:DP\to\mathbf{R}_+$, $(x_1,\ldots,x_n)\to Q(x_1,\ldots,x_n)\in\mathbf{R}_+$ the corresponding production function. The differential of Q is:

$$dQ=\sum_{i=1}^{n}\frac{\partial Q}{\partial x_i}dx_i=\sum_{i=1}^{n}\eta_{x_i}dx_i$$

The maximum output is reached for the level of the factors of production that cancels the first differential of Q, so:

$$\sum_{i=1}^{n} \eta_{x_i} dx_i=0$$

On the other hand, the total cost being constant, we have:

$$CT=p_1x_1+...+p_nx_n=const.$$

from where:

$$0=dCT=p_1dx_1+...+p_ndx_n$$

Considering the system of equations:

$$\begin{cases}\eta_{x_1} dx_1+\ldots+\eta_{x_n} dx_n = 0 \\ p_1 dx_1+\ldots+p_n dx_n = 0\end{cases}$$

we have that if $\exists i\neq j=\overline{1,n}$ such that: $\begin{vmatrix}\eta_{x_i} & \eta_{x_j} \\ p_i & p_j\end{vmatrix}\neq 0$ then, solving the system with the unknowns $dx_1,...,dx_n$, follows that $dx_i$ and $dx_j$ are linear combinations of $dx_k$, $k=\overline{1,n}$, $k\neq i,j$ so the factors of production $x_1,...,x_n$ are not independent which is a contradiction. Thus:

$$\frac{\eta_{x_1}}{p_1}=\ldots=\frac{\eta_{x_n}}{p_n}$$

**Q.E.D.**

***Theorem***

$CTM=\frac{p_1}{w_{x_1}}+\ldots+\frac{p_n}{w_{x_n}}$.

***Proof***

We have: $CTM=\frac{CT}{Q}=\frac{p_1x_1+\ldots+p_nx_n}{Q}=p_1\frac{x_1}{Q}+\ldots+p_n\frac{x_n}{Q}=\frac{p_1}{w_{x_1}}+\ldots+\frac{p_n}{w_{x_n}}$. **Q.E.D.**

From the theorem, it follows that, in particular, considering all factors of production constant except for labor (L), from the fact that average productivity increases up to a certain number of employees after which it decreases, it follows that the average total cost decreases up to a certain number of workers, after which it will increase.

***Theorem***

$Cm=p_1\gamma_{x_1}+\ldots+p_n\gamma_{x_n}$.

***Proof***

We have: $Cm=\frac{\partial CT}{\partial Q}=\frac{\partial(p_1x_1+\ldots+p_nx_n)}{\partial Q}=p_1\frac{\partial x_1}{\partial Q}+\ldots+p_n\frac{\partial x_n}{\partial Q}=p_1\gamma_{x_1}+\ldots+p_n\gamma_{x_n}$. **Q.E.D.**

From the theorem, it follows that, in particular, considering all factors of production constant except for labor (L), from the fact that marginal productivity increases up to a certain number of employees after which it decreases, becoming negative at a sufficiently large number of employees, it results in the inverse variation of the marginal coefficients, so the marginal cost will decrease up to a certain number of workers, after which it will increase.

## 3. Short-Term and Long-Term Costs

We will say that a production process runs in the short run if at least one factor of production remains constant.

In the short run, the only significant variable is work. As a rule, the level of capital, i.e. the totality of the means that contribute to the development of production, is constant, as well as the land, i.e. geographical locations, mineral deposits etc. are constant or approximately constant in the short term. From the analysis of short-term production it follows that, in this case, from a certain number of workers onwards the law of diminishing marginal returns manifests itself.

In the following, we will note, to highlight the special nature of these costs: CVS – short-term variable cost, CTS – short-term total cost, CVMS – short-term average variable cost, CTMS – short-term average total cost , CmS – short-term marginal cost, fixed, quasi-fixed costs or average fixed cost keeping their notations CF, CFM, respectively CCF due to the fact that they can only appear in this situation.

The following relations are absolutely obvious from their definitions:

$$CTS=CF+CVS; \qquad CFM=\frac{CF}{Q}; \qquad CVMS=\frac{CVS}{Q};$$

$$CTMS=\frac{CTS}{Q}=CFM+CVMS; \qquad CmS=\frac{\partial CTS}{\partial Q}=\frac{\partial CVS}{\partial Q}$$

A production process is said to be in the long run if all factors of production are variable. Unlike the previous situation, in long-term production, due to technological changes or improved management, labor productivity can increase. In the long run, no fixed costs are incurred, so we will consider CF=0, CFM=0, and CCF=0.

To make a differentiation in relation to the short term, we will note, in this case: CVL – long-term variable cost, CTL – long-term total cost, CVML – long-term average variable cost, CTML – long-term average total cost long and CmL – long-run marginal cost.

The following relations are also obvious:

$$CTL=CVL; \qquad CVML=\frac{CVL}{Q}; \qquad CTML=\frac{CTL}{Q}=CVML; \qquad CmL=\frac{\partial CTL}{\partial Q}=\frac{\partial CVL}{\partial Q}$$

The long-term total cost CTL is increasing with respect to the level of output Q, a fact resulting from the monotonic property of the cost function with respect to the level of output.

Let us now consider on short term two series of production factors: fixed – $x_1,...,x_k$ with prices $p_1,...,p_k$ and variables: $x_{k+1},...,x_n$ with prices $p_{k+1},...,p_n$.

Let the minimum consumption of fixed factors be: $\bar{x}_1,\dots,\bar{x}_k$ representing the fact that for any quantity consumed by factor $x_i\le\bar{x}_i$, i=$\overline{1,k}$ the related fixed cost will be $p_i\bar{x}_i$. The fixed cost will therefore be:

$$CF=\sum_{i=1}^{k} p_i\bar{x}_i$$

We have therefore:

$$CTS=CF+CVS=\sum_{i=1}^{k} p_i\bar{x}_i+\sum_{j=k+1}^{n} p_j x_j$$

In the long run term, fixed costs naturally turn into variables, achieving:

$$CTL=CVL=\sum_{i=1}^{k} p_i x_i+\sum_{j=k+1}^{n} p_j x_j$$

We obtain therefore:

$$CTL-CTS=\sum_{i=1}^{k} p_i(x_i-\bar{x}_i)$$

If in the short term, the entire amount of fixed factors of production is consumed, we get: $x_i\ge\bar{x}_i$, i=$\overline{1,k}$ therefore: CTL≥CTS and if unconsumed quantities remain of each fixed factor of production, then $x_i\le\bar{x}_i$, i=$\overline{1,k}$ thus: CTL≤CTS. Which of these two situations occurs in practice?

Within a production process, it is obvious that the fixed factors of production are not consumed in greater quantities than they are available. Therefore, we will not reduce the long-run cost to some short-run cost relative to an output that was made at a time when there were no changes in technology or other factors competing to lower costs.

For this reason, we will always have, at a fixed time: $x_i\le\bar{x}_i$, i=$\overline{1,k}$ thus CTL≤CTS. From this formula, we obtain: CTML=$\frac{CTL}{Q}\le\frac{CTS}{Q}$=CTMS.

In any microeconomic theory course, the general forms of short- and long-run costs are presented. The legitimate question is, why do they have these behaviors (graphics)? To clarify this problem we will consider a number of three essential axioms for the behavior of costs, from where we will deduce the well-known forms of their graphs. In the following, we will assume that the cost function is of class $C^3$, that is, it admits derivatives up to order 3 inclusive, and these derivatives are continuous.

**Axiom 1.** The marginal cost CmL / CmS is positive, convex and has a unique local minimum point.

Let $Q^*$ – the unique local minimum point of marginal cost and $CmL_{min}=CmL(Q^*)$. Because $CmL_{min}$ is the minimal value of the function CmL we have: $CmL(Q)\geq CmL_{min}$ for any production Q. On the other hand from the convexity of the function CmL we have that $CmL''(Q)\geq 0$ therefore CmL' is an increasing function. Because $CmL'(Q^*)=0$ we get that $CmL'(Q)<0\ \forall Q<Q^*$ and $CmL'(Q)>0\ \forall Q>Q^*$. But $CmL'=CTL''$ therefore: $CTL''(Q)<0\ \forall Q<Q^*$ - CTL is concave for $Q<Q^*$ and $CTL''(Q)>0\ \forall Q>Q^*$ - CTL is convex for $Q>Q^*$. In the case of the short-term cost, as the derivative of the fixed cost cancels (it being constant) all these statements remain valid. Because of the continuity of the CTL" function, it follows that the inflection point of CTL / CTS is Q* so it coincides with the minimum point of CmL / CmS.

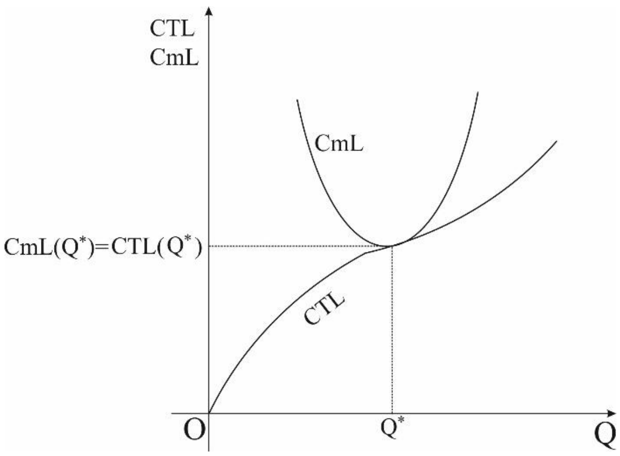


**Figure 1. Long-run Marginal and Total Costs**

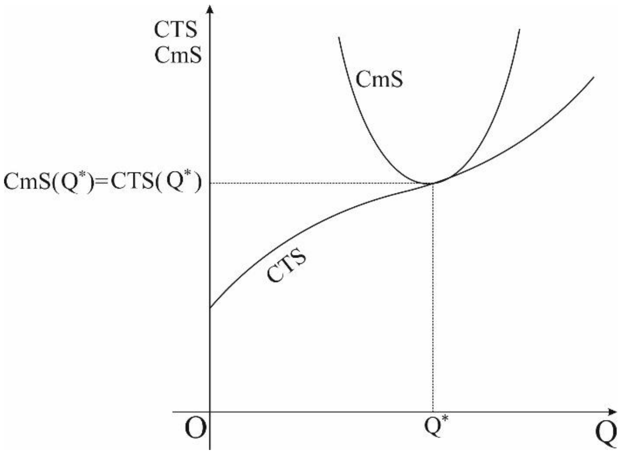


**Figure 2. Short-run Marginal and Total Costs**

**Axiom 2.** The average variable cost CVML / CVMS is positive, convex, and has a unique local minimum point.

We know that: CVML=$\frac{CVL}{Q}$, therefore: CVML'=$\left(\frac{CVL}{Q}\right)'$=$\frac{CVL\prime\cdot Q-CVL}{Q^2}$=$\frac{CVL\prime-CVML}{Q}$= $\frac{CmL-CVML}{Q}$.

Let $Q^{**}$ be the minimal point of CVML that is: CVML'($Q^{**}$)=0 and $CVML_{min}$ the minimal value of CVML. Because $CVML_{min}$ is the minimal value of the function, we have: CVML(Q)≥$CVML_{min}$ for any production Q. On the other hand from the convexity of the function CVML we have that CVML"(Q)≥0 therefore CVML' is an increasing function. Because CVML'($Q^{**}$)=0 we get that CVML'(Q)<0 ∀Q<$Q^{**}$ and CVML'(Q)>0 ∀Q>$Q^{**}$. But from: CVML'=$\frac{CmL-CVML}{Q}$ we obtain that: CmL<CVML ∀Q<$Q^{**}$ and CmL>CVML ∀Q>$Q^{**}$. Also, CmL($Q^{**}$)=CVML($Q^{**}$). As a result of these considerations, it follows that the local minimum point of CVML coincides with the point of intersection of its curve with that of the long-term marginal cost.

For short term costs because CFM=$\frac{CF}{Q}$ we have that CFM is decreasing, CF being constant, therefore CFM'<0 ∀Q>0. But CTMS=CFM+CVMS implies that CTMS'=CFM'+CVMS'<0 ∀Q<$Q^{**}$.

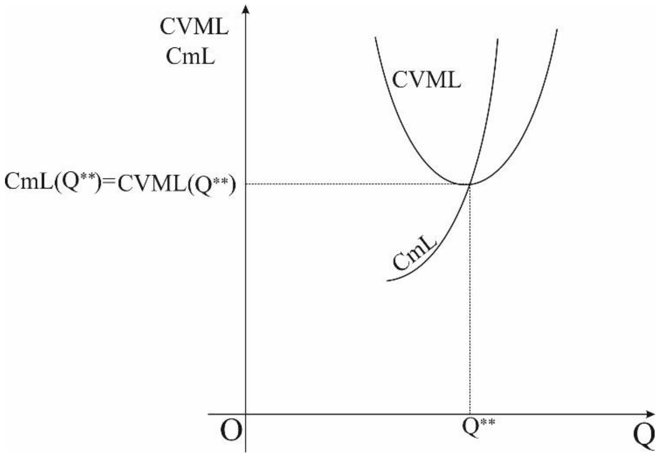


**Figure 3. Marginal and Long-run Average Variable Costs**

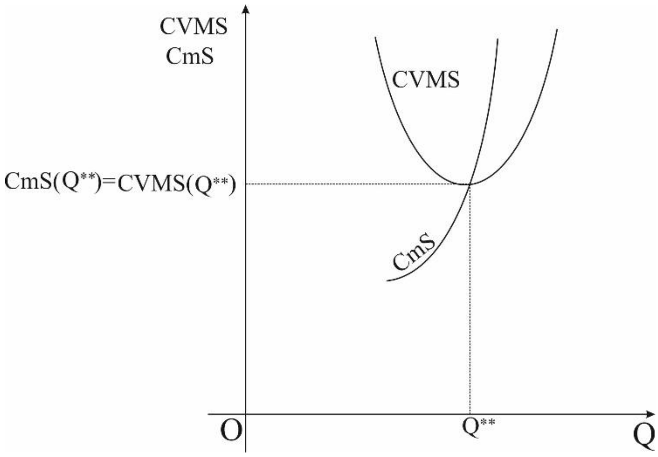


**Figure 4. Marginal and Short-Run Average Variable Costs**

**Axiom 3.** The CTMS short-run average total cost is positive, convex, and increasing for a sufficiently large output value.

From the statement of the axiom 3 ∃Q' such that: CTMS'>0 ∀Q>Q'. how CTMS'<0 ∀Q<$Q^{**}$ implies that $Q^{**}$<Q'. From the continuity of the function CTMS' it follows that ∃Q" such that: CTMS'(Q")=0 therefore Q" is a minimal point. We have therefore Q"∈($Q^{**}$,Q'). We have also:

$$CTMS' = \left(\frac{CTS}{Q}\right)' = \frac{CTS' \cdot Q - CTS}{Q^2} = \frac{CTS' - CTMS}{Q} = \frac{CmS - CTMS}{Q}$$

therefore in the point Q" we have: CmS(Q")=CTMS(Q") and CmS<CTMS ∀Q<Q" and CmS>CTMS ∀Q>Q". From these considerations, it follows that the local minimum point of CTMS coincides with the point of intersection of its curve with that of the short-term marginal cost.

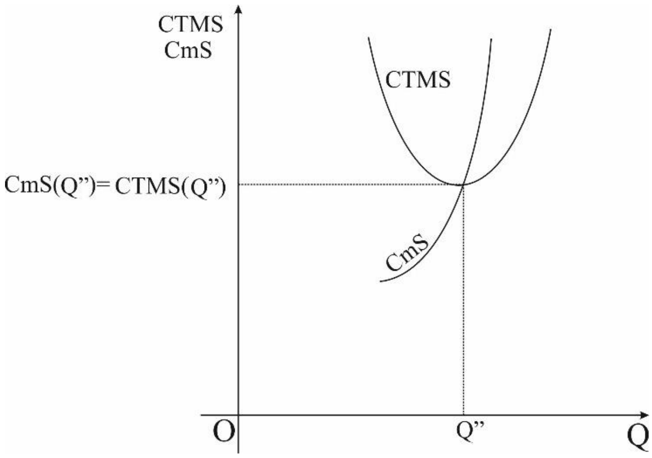


**Figure 5. Marginal and Average Short-Run Total Costs**

From axioms 1-3 we obtained the existence of three points $Q^*$, $Q^{**}$ and Q”. We naturally pose the problem of determining the order of these points for the purpose of plotting the graphs of the above curves.

We successively have: CTMS’=$\frac{CmS-CTMS}{Q}$, CTMS”=$\frac{CmS'Q-2(CmS-CTMS)}{Q^2}$.

Because CTMS is convex, we have that CTMS”($Q^*$)=$-\frac{2(CmS(Q^*)-CTMS(Q^*))}{Q^{*2}}$>0 therefore $CmS(Q^*) < CTMS(Q^*)$. But CmS<CTMS ∀Q<Q” and CmS>CTMS ∀Q>Q” therefore $Q^*$<Q”.

Also, CVMS’=$\frac{CmS-CVMS}{Q}$, CVMS”=$\frac{CmS'Q-2(CmS-CVMS)}{Q^2}$.

In $Q^*$, because CmS’($Q^*$)=0, we have: CVMS”($Q^*$)= $\frac{-2(CmS(Q^*)-CVMS(Q^*))}{Q^{*2}}$, and from convexity of CVMS (axiom 2) follows: CmS($Q^*$)<CVMS($Q^*$) therefore $Q^*<Q^{**}$. From the fact that Q”∈($Q^{**}$,Q’) we have:

$$Q^*<Q^{**}<Q''$$

Because CTMS=CVMS+CFM we have: CTMS>CVMS and CTMS> CFM.

- for $Q<Q_1$: CmS<CVMS<CTMS and CmS↓, CTMS↓, CVMS↓, CFM↓
- for $Q=Q_1$: minimal point of CmS
- for $Q_1<Q<Q_2$: CmS<CVMS<CTMS and CmS↑, CTMS↓, CVMS↓, CFM↓
- for $Q=Q_2$: minimal point of CVMS
- for $Q_2<Q<Q_4$: CVMS<CmS<CTMS and CmS↑, CTMS↓, CVMS↑, CFM↓

- for $Q=Q_4$: minimal point of CTMS
- for $Q_4<Q$: CmS>CTMS>CVMS and CmS↑, CTMS↑, CVMS↑, CFM↓

where ↑ means an increasing function and ↓ a decreasing function.

In addition, CVMS and CTMS curves are convex, and their intersections with CmS occur at the local minimum points of the respective curves. Also, the graph of CFM=CF/Q is an equilateral hyperbola because CFM·Q=CF=constant.

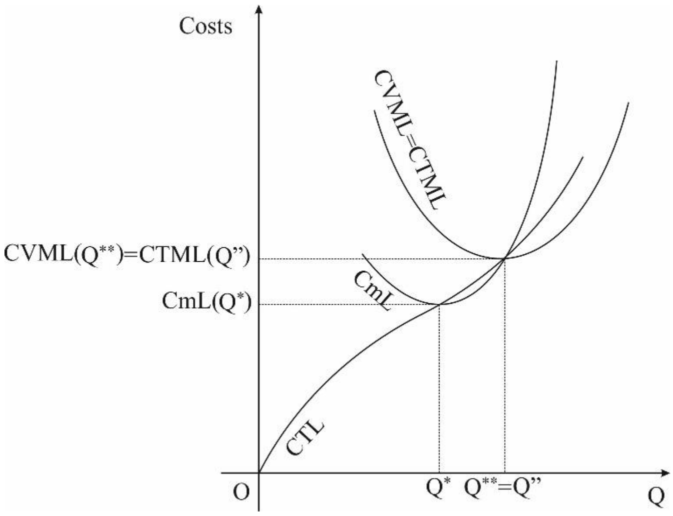


**Figure 6. Long-run Costs**

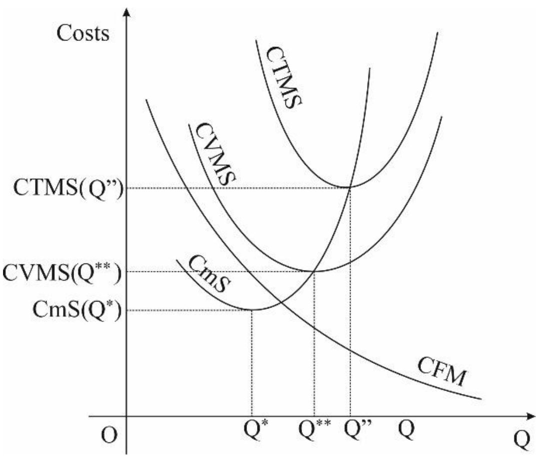


**Figure 7. Short-run Costs**

## 4. Conclusions

The theory of costs is of particular importance in the theory of the producer, giving essential information on the level of production that must be achieved within the limits of acceptable costs in the market. The axiomatic approach to this theory highlights the essential aspects that are rigorously demonstrated and that provide insight into the behavior of different types of costs.

## References

Ioan, G. (2023). The general theory of Costs.to appear in *Acta Universitatis Danubius. Oeconomica.*

Ioan, C.A. & Ioan G. (2022). *Mateconomics*. Galatil: Zigotto Publishing,

Simon, C.P. & Blume, L.E. (2010). *Mathematics for Economists*. W.W.Norton & Company.

Stancu, S. (2006). *Microeconomics*. Bucharest: Ed. Economica.