Exploring Surfaces of Revolution through GeoGebra: A Qualitative Approach to Calculus Teaching

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Abstract: Understanding surfaces of revolution in Calculus often presents challenges due to their abstract and multidimensional characteristics. To address these difficulties, this study investigates the potential of GeoGebra as a dynamic visualization tool to support the teaching and learning of such concepts. **Objectives:** This paper proposes a didactic approach to teaching surfaces of revolution in the Calculus of One Variable course, aiming to enhance students' comprehension of abstract mathematical ideas through visual interaction. It emphasizes the importance of helping students transition from two-dimensional curves to three-dimensional surfaces. **Prior Work:** The study builds on existing research in mathematical visualization use of digital technologies. It contributes to ongoing discussions about incorporating tools like GeoGebra to promote conceptual in mathematics education. **Approach:** A qualitative methodology was employed, involving the design of teaching activities based on formal mathematical definitions and dynamic visualizations using GeoGebra. Results: Integrating GeoGebra can improve students' engagement and spatial reasoning, enabling them to manipulate and better understand the geometry of rotationally generated solids. **Implications:** The results are relevant for educators, researchers, and curriculum developers seeking to enrich mathematics teaching. **Value:** The

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originality of this article lies in the unification of algebraic, geometric, and arithmetic dimensions through GeoGebra to support teaching.

Keywords: Calculus of One Variable; Surfaces of Revolution; Visualization; Geogebra

1. Introduction

The study of surfaces of revolution represents a significant conceptual challenge in higher education, particularly for students in Mathematics, Engineering, and related fields, due to the complexity inherent in transitioning from two-dimensional representations to the construction and analysis of three-dimensional objects generated by the rotation of curves around coordinate axes. This transition demands a developed spatial visualization ability, which is not always sufficiently consolidated among incoming university students, thereby compromising the understanding of essential concepts such as volumes, surface areas, and geometric properties of solids.

In light of these difficulties, the use of interactive digital technologies, especially GeoGebra, emerges as an effective pedagogical alternative capable of overcoming the limitations of traditional resources by offering expanded possibilities for exploration and experimentation. As highlighted by Nadalon and Leivas (2019), GeoGebra enables the construction of dynamic, interactive, and precise visual representations that go beyond mere illustration, constituting a fundamental cognitive instrument mediating between symbolic representations and geometric understanding. This mediating role of visualization is essential for mathematical knowledge construction, as it bridges algebraic and numerical reasoning with spatial and intuitive perception of forms.

Moreover, Alves (2012, 2014) emphasizes that the manipulation of parametric curves and surfaces within computational environments such as GeoGebra and CAS Maple enhances the development of students' topological and geometric intuition, allowing them to perceive complex spatial relationships that would be difficult to apprehend through conventional methods. Such an approach fosters a deeper internalization of concepts, transcending formal understanding to reach qualitative and intuitive levels of knowledge.

Complementing this, the theoretical foundation supporting this proposal is grounded in studies that stress the importance of integrating and articulating multiple representations—numerical, algebraic, and geometric—as an indispensable didactic strategy for effective mathematics teaching (Paiva & Alves, 2018). The didactic transposition of abstract concepts, such as the generation of surfaces by rotation, requires coordination of these diverse forms of representation, ensuring the construction of robust, stable, and enduring meanings that transcend rote memorization of procedures.

In this context, GeoGebra functions as a catalyst for dynamic and interactive teaching, as it enables real-time visualization, manipulation, and analysis of mathematical objects. This not only facilitates conceptual assimilation but also promotes the development of investigative and reflective skills, which are fundamental characteristics of active and critical learning.

Therefore, this work proposes the development of a structured didactic sequence for teaching surfaces of revolution within Single Variable Calculus, exploiting the potentialities of GeoGebra as a tool for construction, investigation, and analysis. The proposal aims to transcend traditional pedagogical approaches, which are centered on passive demonstration and mechanical reproduction of procedures, by establishing an investigative model of teaching in which students actively participate in the educational process through exploration and critical reflection.

Furthermore, this study contributes to the epistemological debate regarding the role of visualization as a didactic and cognitive tool in mathematics education, underscoring the necessity of pedagogical practices that articulate geometric intuition with analytical formalization. Such articulation is fundamental to broadening educators' didactic repertoires and, consequently, enhancing teaching quality in disciplines involving the study of surfaces and solids.

Several studies have demonstrated the relevance of visualization in teaching abstract mathematical concepts, particularly through software such as GeoGebra. Nadalon and Leivas (2019) highlight that this tool enables dynamic and interactive representations of surfaces and solids of revolution, surpassing traditional classroom models in precision and interactivity. Alves (2013a, 2013b, 2014) investigates GeoGebra's potential to support understanding of topological and geometric notions, fostering an integrated and intuitive approach that encourages the construction of profound and lasting meanings.

These investigations reveal that visualization serves not only as a motivating and engaging element for students but also as a crucial link between symbolic manipulation and the development of spatial reasoning. However, it is observed that most pedagogical practices remain limited to demonstrative use of these tools, without fully integrating them into structured didactic sequences. Thus, this article seeks to fill this gap by presenting a carefully planned educational activity that articulates exploration, manipulation, and reflection, grounded in the theoretical contributions of Alves (2012, 2014) and Paiva and Alves (2018), aiming to foster a deeper, investigative, and integrated teaching of surfaces of revolution.

2. Interactive Exploration of Mathematical Concepts Using GeoGebra

The teaching and learning of mathematical concepts have traditionally emphasized the algebraic dimension, focusing mainly on symbolic manipulation, formulaic procedures, and algorithmic reasoning. This emphasis often relegated other essential dimensions of mathematical understanding, namely, the geometric and arithmetic aspects, to a secondary role or, in many cases, left them underexplored in the classroom. As a result, students frequently develop a fragmented understanding centered on symbolic expressions without an integrated grasp of their spatial or quantitative interpretations (Duval, 1993; Paiva & Alves, 2018).

From a qualitative perspective, learning mathematics involves the articulation among three fundamental dimensions: algebraic, geometric, and arithmetic. The algebraic dimension encompasses the manipulation of symbols, the formalization of mathematical relationships, and equation solving. The geometric dimension refers to spatial visualization and the interpretation of mathematical objects, shapes, and their properties. The arithmetic dimension involves numerical reasoning, measurement, and the quantitative evaluation of mathematical phenomena. Together, these dimensions form a triad that supports a more comprehensive and meaningful understanding of mathematical concepts (Alves, 2012; Duval, 1993).

The emergence of dynamic digital technologies, such as GeoGebra, can provide an integrated approach to these dimensions. GeoGebra offers an interactive environment where algebraic expressions, geometric constructions, and numerical values coexist and interact dynamically, allowing learners to explore and connect these different facets simultaneously. For example, representing a function not only symbolically but also as a geometric curve and a table of values encourages coordination between abstract reasoning and concrete intuition (Nadalon & Leivas, 2019; Alves, 2014). Furthermore, Xing, Pei, and Shang (2024) highlight that dynamic teaching through software like GeoGebra promotes greater engagement and understanding in advanced mathematical subjects, especially in digital educational environments.

This integrated visualization is particularly valuable in the qualitative study of mathematical concepts because it promotes the recognition of patterns, behaviors, and properties that might remain hidden in purely symbolic approaches. Through GeoGebra, students can manipulate parameters in real time, observe corresponding changes in graphs and numerical data, and thus engage in an exploratory process that deepens conceptual understanding beyond mechanical procedural knowledge (Paiva & Alves, 2018; Alves, 2013a).

It is important to highlight that the arithmetic dimension, often neglected or superficially addressed in traditional instruction, gains renewed significance in this context. The numerical interpretation of geometric configurations and algebraic relationships becomes accessible and tangible, reinforcing the interconnection of these dimensions. This shift helps to overcome the compartmentalization that previously characterized mathematics education, advancing toward a holistic model in which qualitative insight and quantitative analysis inform and support each other (Duval, 1993; Alves, 2012).

In summary, GeoGebra can provide a qualitative learning environment in which the algebraic, geometric, and arithmetic dimensions are activated and intertwined simultaneously. This approach contrasts with the conventional teaching model centered predominantly on the algebraic dimension, fostering mathematical understanding and expanding students' capacity to visualize, reason, and generalize across multiple representations (Paiva & Alves, 2018; Nadalon & Leivas, 2019; Xing, Pei, & Shang, 2024).

3. Surfaces of Revolution Definitions

A surface of revolution is formed in Euclidean space by rotating a curve (the generatrix) around an axis that typically doesn't intersect the generatrix, except at its endpoints.

Definition: A surface of revolution is generated by rotating a plane curve about a fixed axis. Mathematically, if we have a function defined $y = f(x)$ on an interval $[a, b]$, the surface of revolution obtained by rotating it around the axis can be parameterized as:

$$X(u, v) = (u, f(u)\cos v, f(u)\sin v) \;\; a \le u \le b, \;\; 0 \le v \le 2\pi$$

Similarly, if the rotation occurs around the axis y, the parameterization will be: $X(u, v) = (f(u)\cos v, u, f(u)\sin v) \ \ a \le u \le b, \ \ 0 \le v \le 2\pi$

Examples of surfaces of the revolution:

1.The Sphere generated by rotating the semicircle $f(u) = \sqrt{r^2 - u^2}$, with $-r \le u \le r$:

$$X(u, v) = \left(u, \sqrt{r^2 - u^2} \cos v, \sqrt{r^2 - u^2} \sin v\right)$$

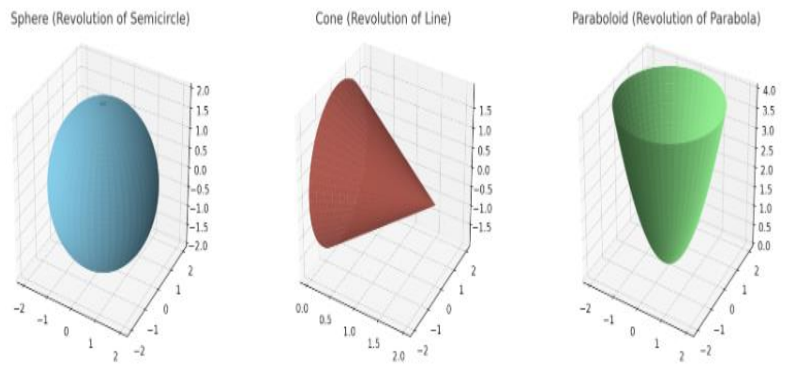
2. Cone generated by rotating the line $f(u) = ku$, with $0 \le u \le h$:

$$X(u, v) = (u, ku\cos v, ku \sin v)$$

3.Paraboloid generated by rotating the parabola $f(u) = u^2$, with $0 \le u \le R$:

$$X(u, v) = (u, u^2\cos v, u^2 \sin v)$$

Figure 1. Shows the three surfaces of revolution: the sphere, the cone, and the paraboloid, respectively



4. Designing a Didactic Situation for Understanding Solids of Revolution

To enhance conceptual understanding and promote visual reasoning, we propose the design of a didactic situation centered on the geometric and analytical exploration of solids of revolution.

4.1. 1st Didactic Situation

Consider the analytical and graphical study of the curve $f(u) = \sin(u)$ defined on the interval $u \in [0,2\pi]$ and the corresponding surfaces of revolution generated by rotating this curve around the coordinate axes.

The rotation around the **x-axis** results in a symmetric surface resembling a sequence of tubular waveforms. When rotated around the **y-axis**, the construction requires a reparametrization of the original function, leading to a surface with distinct geometric features. In the case of rotation around the **z-axis**, the surface is obtained through a suitable representation in cylindrical coordinates.

The visualization of the resulting surfaces will be carried out using GeoGebra, with the aim of integrating symbolic analysis and spatial reasoning to enhance the geometric understanding of the concepts involved.

Step 1: Construction of the base curve the curve $f(x) = \sin(x)$ defined on the interval $x \in [0,2\pi]$

Step 2: Rotation around the x-axis When rotating the curve around the x-axis, each point (x, y, 0) generates a circle in the plane perpendicular to the x-axis, in the yz-plane.

The surface can be parameterized by:

$\mathrm{X(u,v)} = (\mathrm{u}, \sin(\mathrm{u})\cos \mathrm{v}, \sin(\mathrm{u})\sin \mathrm{v}), u \in [0,2\pi], v \in [0,2\pi]$.

In GeoGebra, you can create a movable point $P = (x, \sin(x), 0)$ and use the 3D rotation command to rotate P around the x-axis by the angle v.

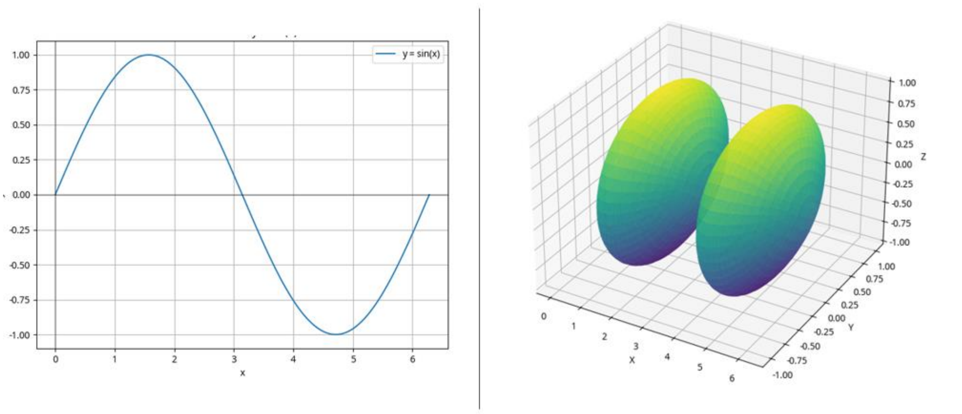


Figura 2. Surface of Revolution Around the X-Axis

Step 3: Rotation around the y-axis Rotating the curve around the y-axis, the parameterization is:

$\mathrm{X(u,v)} = (\mathrm{u}\cos(\mathrm{v}), \sin(\mathrm{u}), \mathrm{u}\sin(\mathrm{v})), u \in [0,2\pi], v \in [0,2\pi]$.

In GeoGebra, it is possible to configure the curve in the xy-plane and apply the rotation around the y-axis.

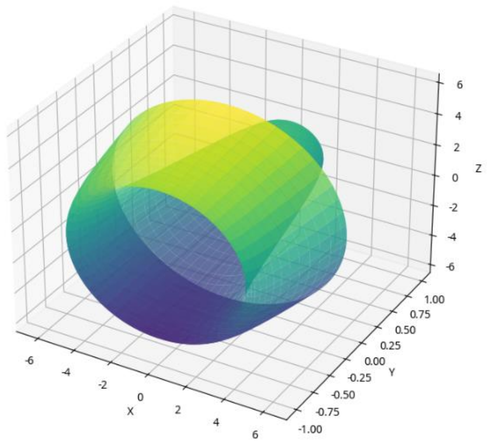


Figura 3. Surface of revolution Around the Y-Axis

Step 4: Rotation around the z-axis Here the curve will initially be positioned in the xz-plane, for example, (x, 0, sin(x)).

The parameterization of the surface by rotation around the z-axis is:

$\mathrm{X(u,v)} = (\mathrm{u}\cos(\mathrm{v}), \mathrm{u}\sin(\mathrm{v}), \sin(\mathrm{u})), u \in [0,2\pi], v \in [0,2\pi]$.

This rotation is less intuitive visually because the curve "spreads" in three-dimensional space in a complex way.

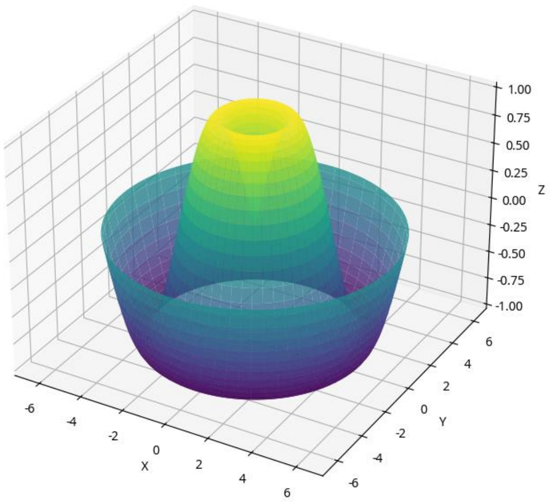


Figura 4. Surface of Revolution Around the Z-Axis

4.2. 2st Didactic Situation:

Let the function $f(x) = \cos(x) + x$, we will analyze with the help of geogebra the behavior of the surfaces generated when rotated in relation to the x, y and z axes. On which surfaces will you have the following expressions:

$\text{Superfície}^1 = (\mathrm{x}, \cos(\beta)\ \mathrm{f(x)}, \sin(\beta)\ \mathrm{f(x)})$with $0\ \leq x \leq 4$

$\text{Superfície}^2 = (\sin(\beta)\ \mathrm{f(y)}, \mathrm{y}, \cos(\beta)\ \mathrm{f(y)})$ with $0\ \leq y \leq 4$

$\text{Superfície}^3 = (\cos(\beta)\mathrm{f(z)}, \sin(\beta)\mathrm{f(z)}, \mathrm{z})$ with $0\ \leq z \leq 4$,

Where is the angle of rotation (from 0 to 2π).

Rotating a curve around an axis generates a three-dimensional surface where each point on the curve describes a circular path in a plane perpendicular to the chosen axis. Depending on the axis of rotation, the resulting shape may present different symmetries, oscillations and variations in curvature.

Below, in figure 5, we present a detailed analysis of each of these rotations, highlighting their characteristics and interpreting the results based on the associated mathematical expressions

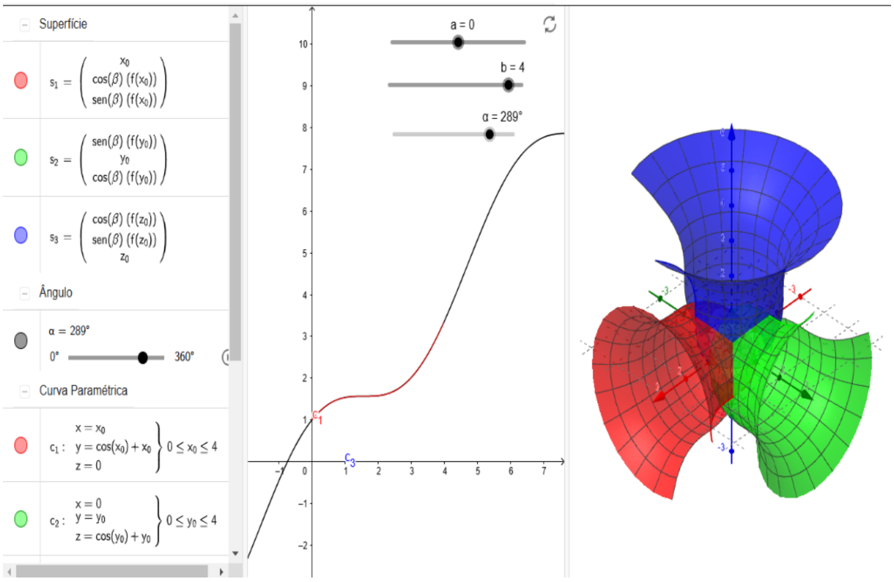


Figura 5. Surface 1(red), surface 2(green) and surface 3(blue).

Using GeoGebra, it is possible to visualize these surfaces with precision by inputting the corresponding parametric equations and simulating the rotation of the curve around each of the coordinate axes. This dynamic visualization enables the observation of the geometric properties of the resulting surfaces, including the influence of the oscillatory behavior of the function cos(x) on their structure. GeoGebra provides an effective means to analyze how each axis of rotation affects the symmetry, curvature, and overall shape of the generated surface, thereby enhancing the conceptual understanding of three-dimensional transformations.

5. Conclusion

This study highlighted the relevance of using digital technologies in the approach to complex mathematical topics such as surfaces of revolution. Based on the design of didactic situations involving the construction and manipulation of parametric models in a dynamic environment, a set of activities is proposed to foster conceptual and visual exploration of these topics, overcoming the limitations of purely analytical or static methods.

The proposal includes the use of GeoGebra as an interactive visualization tool for the qualitative analysis of surfaces generated by rotations around the coordinate axes. This approach aims to promote the articulation between different registers of representation — algebraic, graphical, and geometric — and to stimulate the development of spatial and geometric reasoning.

The didactic situations formulated are intended to contribute to the construction of more effective pedagogical strategies in mathematics education, exploring the potential of digital technologies to mediate between formal abstraction and visualization. It is expected that, when implemented in educational contexts, these proposals may enrich the teaching of advanced mathematical topics and support more innovative practices aligned with the contemporary demands of the classroom.

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